Ya. I. Kanel'

1.** Consider the Cauchy problem for the equation

1960

SovietRxiv

Abstract

Full Text

Ya. I. Kanel'

ON THE BEHAVIOR OF SOLUTIONS OF THE CAUCHY PROBLEM AS TIME INCREASES WITHOUT BOUND FOR QUASILINEAR EQUATIONS OCCURRING IN THE THEORY OF COMBUSTION

(Presented by Academician I. G. Petrovskii, January 12, 1960)

1. Consider the Cauchy problem for the equation

$$\partial u/\partial t - \partial^2 u/\partial x^2 = F(u) \tag{1}$$

with the initial condition

$$u\big|_{t=0} = 0 \quad \text{for } x < 0; \qquad u\big|_{t=0} = 1 \quad \text{for } x > 0. \tag{2}$$

$F(u)$ is defined and continuously differentiable for $0 \leq u \leq 1$, and

$$F(0) = F(1) = 0. \tag{3}$$

For the theory of combustion, of interest is the behavior of the solution of problem (1)–(2) as $t \to \infty$ in the case where $F(u)$ satisfies conditions (3) and

$$F(u) \equiv 0 \quad \text{for } 0 < u < \alpha < 1, \quad \text{where } \alpha \text{ is some number}; \tag{4}$$

$$F(u) > 0 \quad \text{for } \alpha < u < 1; \qquad F'(1) < 0. \tag{5}$$

The existence and uniqueness theorem for problem (1)–(2) was proved in ([1]). The behavior of the solution of problem (1)–(2) as $t \to \infty$ was also investigated there, when $F(u)$ satisfies conditions (3) and $F'(0) > 0$, $F'(u) < F'(0)$ for $0 < u \leq 1$, and $F(u) > 0$ for $0 < u < 1$.

A general formulation of the question of the behavior of the solution of the Cauchy problem for large t for quasilinear systems is given in ([4]). This question is also considered in works ([2,3]), and in ([3]) the most general results were obtained for the equation $\partial u/\partial t - \partial^2 u/\partial x^2 + \partial\varphi(u)/\partial x = 0$. A solution of equation (1) of the form $\tilde{u}(x + mt + C)$, $C = \text{const}$, $-\infty < C < +\infty$, satisfying the conditions

$$\lim_{x\to-\infty} \tilde{u} = 0, \qquad \lim_{x\to+\infty} \tilde{u} = 1 \tag{6}$$

will be called stationary.

In the present work the following conditions are imposed on $F(u)$: (3), (5), and, instead of (4), the more general condition

$$F(u) \le 0 \quad \text{for } 0 < u < \alpha < 1; \qquad F'(u) \le 0 \quad \text{for } 0 < u < \alpha_1 \le \alpha;$$

$$\int_0^1 F(u)\,du > 0. \tag{7}$$

Analogously to ([4,5]), it can be shown that under our restrictions on $F(u)$ there exists a stationary solution, unique up to a shift C along the x-axis.

Theorem. Let $u(x,t)$ be a solution of problem (1)–(2), where $F(u)$ satisfies conditions (3), (5), (7). Then, uniformly in all x, $-\infty < x < +\infty$,

$$|\tilde{u}(x + mt + C_0) - u(x,t)| \to 0 \quad \text{as } t \to \infty,$$

where C_0 is a certain constant.

The requirement that $F'(u)$ be nonpositive in neighborhoods of $u = 0$ and $u = 1$ is quite essential. It is precisely this that ensures, uniformly in all x, $-\infty < x < +\infty$, the closeness of $u(x,t)$ to some stationary solution for large t, which is absent in the case considered in (1), where $F'(0) > 0$.

2. Proof of the theorem. Pass to the variables $x' = x + mt$, $t' = t$, and denote them again by x and t. Equation (1) then takes the form

$$\partial u/\partial t - \partial^2 u/\partial x^2 + m\,\partial u/\partial x = F(u), \tag{8}$$

and the stationary solution will depend only on x.

As in ([4,5]), one can show that in the plane of the variables $u = \tilde{u}$, $p = \tilde{u}'(x+C)$, the stationary solution corresponds to a curve whose equation is $p = p(u)$, where

$$p(0) = p(1) = 0; \qquad p(u) > 0 \quad \text{for } 0 < u < 1; \qquad p'(0) > 0,\ p'(1) < 0. \tag{9}$$

The following lemmas 1, 2, and 3 were proved in (1) for the solution of problem (1)–(2), but are trivially transferred to the solution $u(x,t)$ of problem (8)–(2).

Lemma 1. For $t > 0$ we have

$$0 < u < 1, \qquad \partial u/\partial x > 0. \tag{10}$$

Consider the quantity $\partial u/\partial x = \psi(u,t)$, where $t > 0$, as a function of u and t. This is possible by Lemma 1.

Lemma 2. As t increases and u is fixed, the function ψ does not increase, and

$$\partial u/\partial x = \psi(u,t) \geq p(u) > 0 \quad \text{for } 0 < u < 1,\ t > 0. \tag{11}$$

Let $x = x(t,c)$ be the equation of some level line of the solution $u(x,t)$, i.e. $u[x(t,c),t] = c = \text{const}$, $0 < c < 1$. Put $u^*(\xi,t) = u[\xi + x(t,c),t]$.

Lemma 3. Uniformly in all ξ, $-\infty < \xi < +\infty$, $u^*(\xi,t) \to u^*(\xi)$ as $t \to \infty$, where $u^*(\xi)$ increases strictly monotonically, uniformly continuously in all ξ, and

$$\lim_{\xi\to-\infty} u^*(\xi) = 0, \qquad \lim_{\xi\to-\infty} u^*(\xi) = 1. \tag{12}$$

Lemma 4. As $|x| \to \infty$, $\partial u/\partial x \to 0$, and for arbitrarily large $K > 0$,

$$u(x,t)/e^{Kx} \to 0 \quad \text{as } x \to -\infty, \qquad (1-u)/e^{-Kx} \to 0 \quad \text{as } x \to +\infty.$$

The convergence is uniform in t on any interval $0 < \eta \leq t \leq T < +\infty$.

The lemma is proved by means of an explicit representation of the solution in terms of the initial function and the right-hand side of the equation, using condition (3).

Choose numbers $0 < \alpha_0 < \beta_0 < 1$ such that $F'(u) \leq 0$ for $0 < u < \alpha_0$ and $\beta_0 < u < 1$, and $\tilde{u}'' > 0$ for $\tilde{u} < \alpha_0$, $\tilde{u}'' < 0$ for $\tilde{u} > \beta_0$. This is possible on the basis of the restrictions on $F(u)$ and (9).

Let $\tilde{u}[x + C(x_0,t)]$ be a stationary solution, where the constant $C = C(x_0,t)$ is chosen so that, for the given x_0 and $t > 0$,

$$\tilde{u}[x_0 + C(x_0,t)] = u(x_0,t). \tag{13}$$

Lemma 5. For any prescribed x_0 and $t > 0$ such that $u(x_0,t) < \alpha_0$ ($u(x_0,t) > \beta_0$), the inequality

$$u(x,t) \leq \tilde{u}[x + C(x_0,t)] \quad \text{for } x \leq x_0 \quad (u(x,t) \geq \tilde{u}[x + C(x_0,t)] \quad \text{for } x \geq x_0) \tag{14}$$

holds.

The proof follows from (11), the monotonicity of u and $\tilde{u}$ with respect to x, Lemma 4, and the strict increase of $\tilde{u}'$ with respect to x for $\tilde{u} < \alpha_0$ (strict decrease for $\tilde{u} > \beta_0$).

Lemma 6. For any prescribed x_0 and $\eta > 0$ there exists a constant $\gamma = \gamma(x_0,\eta)$, $0 < \gamma < 1/2$, depending only on x_0 and η, such that for all $t \geq \eta$

$$\gamma < u(x_0,t) < 1 - \gamma. \tag{15}$$

Proof. Let $u(x_0,t) < \alpha_0$, $t \geq \eta$. From equation (8) and Lemma 4 we have

$$\frac{d}{dt}\int_{-\infty}^{x_0} u(x,t)\,dx = \frac{\partial u}{\partial x}(x_0,t) - mu(x_0,t) + \int_{-\infty}^{x_0} F(u)\,dx. \tag{16}$$

From equation (8) for $\tilde{u}$ and the equalities $\lim \tilde{u} = \lim \tilde{u}' = 0$ as $x \to -\infty$ (see (6) and (9)) we have

$$0 = \tilde{u}'[x_0 + C(x_0,t)] - m\tilde{u}[x_0 + C(x_0,t)] + \int_{-\infty}^{x_0} F(\tilde{u})\,dx. \tag{17}$$

Subtracting (17) from (16), using (11), (13), Lemma 5, the monotonicity of $\tilde{u}$ in x, and the inequality $F'(u) \le 0$ for $u \le \alpha_0$, we obtain

$$\frac{d}{dt}\int_{-\infty}^{x_0} u(x,t)\,dx \ge 0. \tag{18}$$

If $u(x_0,t) > \beta_0,\ t \ge \eta$, then analogously one can obtain that

$$\frac{d}{dt}\int_{x_0}^{+\infty} [1 - u(x,t)]\,dx \ge 0. \tag{19}$$

Let $u(x_0,t) \ge \alpha_0,\ t \ge \eta$. Then we obtain, taking into account that for $t \ge \eta$, by Lemmas 4 and 2, $\partial u/\partial x = \psi(u,t) \le \psi(u,\eta) \le M < +\infty$,

$$\int_{-\infty}^{x_0} u(x,t)\,dx \ge \frac{\alpha_0^2}{2M}. \tag{20}$$

Analogously, when $u(x_0,t) \le \beta_0,\ t \ge \eta$,

$$\int_{x_0}^{+\infty} (1-u)\,dx \ge \frac{(1-\beta_0)^2}{2M}. \tag{21}$$

By Lemma 1,

$$\int_{-\infty}^{x_0} u(x,\eta)\,dx > 0, \qquad \int_{x_0}^{+\infty} [1 - u(x,\eta)]\,dx > 0. \tag{22}$$

From (18)–(22) follows the existence of a positive constant $\gamma_1 = \gamma_1(x_0,\eta)$, depending only on x_0 and η, such that for all $t \ge \eta$

$$\int_{-\infty}^{x_0} u(x,t)\,dx > \gamma_1, \qquad \int_{x_0}^{+\infty} [1 - u(x,t)]\,dx > \gamma_1. \tag{23}$$

Taking into account (11), (9) and the monotonicity of u in x, one can prove that for some constant $q > 0$

$$u(x,t) < \alpha_0 e^{q(x-x_0)} \quad \text{when } u(x_0,t) < \alpha_0,\ x \le x_0; \tag{24}$$

$$1-u(x,t)<(1-\beta_0)e^{-q(x-x_0)} \quad \text{when } u(x_0,t)>\beta_0,\ x\geq x_0. \tag{25}$$

The assertion of the lemma follows from relations (23)–(25).

Lemma 7. For any level line defined by the equation $x=x(t,c)$, $0<c<1$, the function $x=x(t,c)$ is uniformly bounded for all $t\geq\eta>0$.

Proof. For the given x_0 and $\eta>0$, define, by Lemma 6, $\gamma=\gamma(x_0,\eta)$. In view of the inequality $\partial u/\partial x>0$ for $t>0$ and (15),

$$x(t,\gamma/2)<x_0<x(t,1-\gamma/2). \tag{26}$$

From the equation $\partial u/\partial x=\psi(u,t)$ we have

$$x(t,\gamma/2)-\int_c^{\gamma/2}\frac{dv}{\psi(v,t)}=x(t,c)=x(t,1-\gamma/2)-\int_c^{1-\gamma/2}\frac{dv}{\psi(v,t)}. \tag{27}$$

In view of (11), the integrals in (27) are bounded uniformly in $t\geq\eta$, and (26), together with (27), gives the assertion of the lemma.

Lemma 8. There exists a finite limit $\lim x(t,c)$ as $t\to\infty$, $0<c<1$.

Proof. According to Lemma 7 and the monotonicity of u with respect to x, one can choose x_0 so that for all $t\geq\eta>0$ and $x\leq x_0$, $u(x,t)<\alpha_0$. Then the inequalities (18) and (24) hold, ensuring the existence of the finite limit

$$\lim_{t\to\infty}\int_{-\infty}^{x_0}u(x,t)\,dx.$$

Let $t_n\to\infty$ be any sequence for which there exists a finite limit $x(t_n,c)$ as $n\to\infty$. By Lemma 3,

$$u(x,t_n)=u^*[x-x(t_n,c),t_n]\to u^*\Big[x-\lim_{n\to\infty}x(t_n,c)\Big] \quad \text{as } n\to\infty; \tag{28}$$

the convergence is uniform in all x. Taking also (24) into account, we obtain

$$\lim_{t\to\infty}\int_{-\infty}^{x_0}u(x,t)\,dt=\lim_{n\to\infty}\int_{-\infty}^{x_0}u(x,t_n)\,dx=\int_{-\infty}^{x_0}u^*\Big[x-\lim_{n\to\infty}x(t_n,c)\Big]\,dx. \tag{29}$$

The existence of the finite limit $\lim x(t,c)$ as $t\to\infty$ now follows directly from Lemma 7, (28), the strict increase of u^*, and (29).

To prove the theorem, fix the level line $x = x(t, c_0)$, $0 < c_0 < 1$. By Lemmas 3 and 8,

$$u(x,t) = u^*[x - x(t, c_0), t] \to u^*(x - C_0)$$

as $t \to \infty$, uniformly in all x, where $C_0 = \lim x(t, c_0)$, $t \to \infty$. Considering $u^*(x - C_0)$ as the limit, as $t_0 \to \infty$, of the sequence of solutions $u_{t_0} = u(x, t_0 + t)$ of equation (8), we obtain that $u^*(x - C_0)$ is a solution of equation (8) in the sense of the integral identity (([6]), §5). Taking into account also that, by Lemmas 2 and 4, for any $x_1 \neq x_2$,

$$0 \leq \frac{u^*(x_1 - C_0) - u^*(x_2 - C_0)}{x_1 - x_2} = \lim_{t \to \infty} \frac{u(x_1, t) - u(x_2, t)}{x_1 - x_2} < +\infty,$$

we may assert, on the basis of ([6]), §5, that $u^*(x - C_0)$ satisfies equation (8) in the classical sense. Passing to the original variables and taking (12) into account, we obtain the assertion of the theorem.

Remark. On $F(u)$ one may impose, for example, the following restrictions:

$$F'(0) < 0; \quad F'(1) < 0; \quad F(0) = F(1) = 0; \quad \int_0^1 F(v)\,dv = 0; \quad \int_u^1 F(v)\,dv > 0$$

for $0 < u < 1$. In this case $m = 0$.

I express my deep gratitude to Prof. O. A. Oleinik for her constant help in this work.

Moscow State University
named after M. V. Lomonosov

Received
11 I 1960

CITED LITERATURE

1. A. N. Kolmogorov, I. G. Petrovsky, N. S. Piskunov, *Bull. Moscow State Univ.*, Section A, 1, no. 6 (1937).

2. E. Hopf, *Comm. on Pure and Appl. Math.*, 3, no. 3, 201 (1950).

3. A. M. Ilyin, O. A. Oleinik, *Dokl. Akad. Nauk SSSR*, 120, no. 1, 25 (1958).

4. I. M. Gelfand, *Uspekhi Mat. Nauk*, 14, issue 2 (86), 87 (1959).

5. Ya. B. Zel' dovich, *Zh. Fiz. Khim.*, 22, no. 1, 27 (1948).

6. O. A. Oleinik, *Uspekhi Mat. Nauk*, 12, issue 3 (75), 3 (1957).

Note: Figure translations are in progress. See original paper for figures.