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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

**M. S. BRODSKII**

## ON THE TRIANGULAR REPRESENTATION OF CERTAIN OPERATORS WITH COM- PLETELY CONTINUOUS IMAGINARY PART

*(Presented by Academician V. I. Smirnov on 16 IV 1960)*

Let  $A$  be a bounded operator with purely real spectrum, acting in a Hilbert space  $H$ . If  $P$  is some projector in  $H$ , then by  $A_P$  we shall denote the operator  $PAP$ , considered in the subspace  $PH$ , and by  $\sigma(A_P)$  the spectrum of the operator  $A_P$ . We shall say that the projector  $P$  **cuts the spectrum of the operator**  $A$  at the point  $t$  if the subspace  $PH$  is invariant with respect to  $A$  and if  $\sigma(A_P) \subset (-\infty, t]$ ,  $\sigma(A_{E-P}) \subset [t, \infty)$ .

We assign the operator  $A$  to the class  $\Omega$  if:

- I. The imaginary part  $K = \frac{A - A^*}{2i}$  of the operator  $A$  is completely continuous.
- II. The entire spectrum of the operator  $A$  lies on the real axis.
- III. There exists a system of projectors  $P_t$  ( $-\infty < t < \infty$ ) such that  $P_{t_1} \leq P_{t_2}$  for  $t_1 < t_2$ , and each projector  $P_t$  cuts the spectrum of the operator  $A$  at the point  $t$ .

From results of Yu. I. Lyubich and V. I. Matsaev <sup>(1)</sup> it follows that every operator  $A$  satisfying conditions I and II, as well as the condition

$$\int_0^\varepsilon \ln^+ \ln^+ M(\delta) d\delta < \infty \quad \left( M(\delta) = \sup_{|\operatorname{Im} \lambda| \geq \delta} \|(A - \lambda E)^{-1}\| \right) \quad (1)$$

belongs to the class  $\Omega$ . In particular, as V. I. Matsaev showed, condition (1) will be fulfilled if the series

$\sum_{n=1}^\infty \frac{|\omega_n|}{n}$  converges, where  $\omega_n$  are the eigenvalues of the operator  $K$ , numbered with multiplicities taken into account in decreasing order of their absolute values.

In the present article the problem of reducing operators of the class  $\Omega$  to triangular form is solved.

1. Let the operator  $A$  belong to the class  $\Omega$ . Then on some closed set of real numbers  $\mathfrak{M}$  ( $\min \mathfrak{M} = 0$ ,  $\max \mathfrak{M} = 1$ ) there exists a function  $E(x)$

$(x \in \mathfrak{M})$  having the following properties <sup>(2)</sup>:

- 1) the values of the function  $E(x)$  are projectors in  $H$ ;
- 2) if  $x_1 < x_2$ , then  $E(x_1) < E(x_2)$ ;
- 3)  $E(0) = 0$ ,  $E(1) = E$ ;
- 4) the function  $E(x)$  is continuous on  $\mathfrak{M}$ ;
- 5) each subspace  $E(x)H$  is invariant with respect to  $A$ ;
- 6) the set of values of the function  $E(x)$  contains all projectors  $P_t$  ( $-\infty < t < \infty$ );
- 7) if  $(a, b)$  is an interval complementary to the set  $\mathfrak{M}$ , then there does not exist a subspace  $H_0$  invariant with respect to  $A$  and satisfying the condition  $E(a)H \subset H_0 \subset E(b)H$ .

Let us note <sup>(3)</sup> that if  $P$  and  $P'$  are projection operators onto subspaces invariant with respect to  $A$ , and  $P < P'$ , then, by virtue of condition 1),

$$\sigma(A_{P'}) = \sigma(A_P) + \sigma(A_{P'-P}), \quad \sigma(A_{E-P}) = \sigma(A_{E-P'}) + \sigma(A_{P'-P}). \quad (2)$$

The first of these equalities shows that the function  $\alpha(x) = \max \sigma(A_{E(x)})$  is nondecreasing. Introduce the function  $\alpha(x+0)$ , defining it at the point  $x$  in the usual way if every interval  $(x, x+\varepsilon)$  contains points of the set  $\mathfrak{M}$ , and putting  $\alpha(x+0) = \alpha(x)$  otherwise.

We shall need the following properties of the functions  $E(x)$  and  $\alpha(x)$ .

- a) The projector  $E(x)$  splits the spectrum of the operator  $A$  at the point  $\alpha(x)$ .
- b) The function  $\alpha(x)$  is continuous from the left.
- c) If  $x < x'$  ( $x, x' \in \mathfrak{M}$ ), then the spectrum of the operator  $A_{E(x')-E(x)}$  lies in the segment  $[\alpha(x+0), \alpha(x')]$ .

Indeed, if  $\lambda \in \sigma(A_{E(x')-E(x)})$ , then, by virtue of (2),  $\lambda \in [\alpha(x), \alpha(x')]$ . Let there exist a strictly decreasing sequence  $\{x_n\} \subset \mathfrak{M}$  tending to  $x$ . Then  $\alpha(x_n) \geq \alpha(x+0)$  and, consequently, for any  $\varepsilon > 0$ ,

$$E(x) = \lim_{n \rightarrow \infty} E(x_n) \geq P_{\alpha(x+0)-\varepsilon}.$$

Therefore

$$\lambda \in \sigma(A_{E-E(x)}) \subset \sigma(A_{E-P_{\alpha(x+0)-\varepsilon}}) \subset [\alpha(x+0) - \varepsilon, \infty),$$

i.e.  $\lambda \in [\alpha(x+0), \infty)$ .

d) If  $(a, b)$  is a complementary interval of the set  $\mathfrak{M}$ , then

$$\dim\{E(b)H \ominus E(a)H\} = 1, \quad (3)$$

and

$$(E(b) - E(a))A(E(b) - E(a)) = \alpha(b)(E(b) - E(a)). \quad (4)$$

In fact,  $\sigma(A_{E(b)-E(a)}) \subset [\alpha(a), \alpha(b)]$ . Let  $\lambda$  belong to the spectrum of the operator  $A$  and let  $\alpha(a) < \lambda < \lambda_1 < \alpha(b)$ . Since some projector  $E(x)$  splits the spectrum of the operator  $A$  at the point  $\lambda_1$ , we have  $\lambda \leq \alpha(x) \leq \lambda_1$ , which is impossible. Thus the set  $\sigma(A_{E(b)-E(a)})$  consists either of the two points  $\alpha(a)$  and  $\alpha(b)$ , or of the single point  $\alpha(b)$ . If the first case occurs, then, by a well-known theorem of Riesz, the operator  $A_{E(b)-E(a)}$  has a nontrivial invariant subspace  $H_0$ . But then the subspace  $H_1 = E(a)H \oplus H_0$  is invariant with respect to  $A$ , and  $E(a)H \subset H_1 \subset E(b)H$ , which contradicts property 7) of the function  $E(x)$ . Thus the operator  $A_{E(b)-E(a)}$  has only one spectral point  $\alpha(b)$ . Since its imaginary part is completely continuous, it has the form  $\alpha(b)E + B$ , where  $B$  is a completely continuous operator<sup>(4)</sup>. If we suppose that the subspace  $E(b)H \ominus E(a)H$  is not one-dimensional, then the operator  $A_{E(b)-E(a)}$  will have a nontrivial invariant subspace by the theorem of Aronszajn and Smith<sup>(5)</sup>, which again leads to a contradiction.

2. We shall say that the points  $0 = x_0 < x_1 < \dots < x_n = 1$  ( $x_k \in \mathfrak{M}$ ) form a  $\delta$ -partition of the set  $\mathfrak{M}$ , if  $x_{k-1}$  and  $x_k$  ( $k = 1, 2, \dots, n$ ) are either endpoints of an interval complementary to  $\mathfrak{M}$ , or satisfy the condition  $x_k - x_{k-1} < \delta$ . It is easy to verify that the integral

$$\int_{\mathfrak{M}} \alpha(x) dE(x) = \lim_{\delta \rightarrow 0} \sum_{k=1}^n \alpha(\xi_k) \Delta E_k \quad (x_{k-1} < \xi_k \leq x_k, \xi_k \in \mathfrak{M}, \Delta E_k = E(x_k) - E(x_{k-1}))$$

converges in norm in the sense of S. O. Shatunovskii. We also note that, by virtue of (4),

$$\lim_{\delta \rightarrow 0} \left\| \sum_{k=1}^n \Delta E_k \frac{A - A^*}{2i} \Delta E_k \right\| = 0.$$

**Lemma.** For every  $\varepsilon > 0$  there exists a partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the set  $\mathfrak{M}$  such that

$$\left\| \sum_{k=1}^n \Delta E_k A \Delta E_k - \int_{\mathfrak{M}} \alpha(x) dE(x) \right\| < \varepsilon \quad (\Delta E_k = E(x_k) - E(x_{k-1})).$$

**Proof.** Let  $\varepsilon > 0$ . There exists a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  of the set  $\mathfrak{M}$  such that  $t_{k-1}$  and  $t_k$  ( $k = 1, 2, \dots, m$ ) are either endpoints of an interval complementary to  $\mathfrak{M}$ , or satisfy the condition  $\alpha(t_k) - \alpha(t_{k-1} + 0) < \varepsilon/4$ . Moreover, there exists a constant  $M$  such that

$$\left\| \left( A_{E(t_k) - E(t_{k-1})} - \lambda E \right)^{-1} \right\| < M$$

$$(k = 1, 2, \dots, m; \operatorname{Im} \lambda = 0, \quad \lambda \bar{\in} [\alpha(t_{k-1} + 0) - \varepsilon/4, \alpha(t_k) + \varepsilon/4]).$$

Let  $0 = x_0 < x_1 < \dots < x_n = 1$  be a refinement of the above partition satisfying the conditions

$$\left\| \sum_{k=1}^n \Delta E_k \frac{A - A^*}{2} \Delta E \right\| < \min \left\{ \frac{1}{M}, \frac{\varepsilon}{4} \right\} \quad (\Delta E_k = E(x_k) - E(x_{k-1})), \quad (5)$$

$$\left\| \sum_{k=1}^n \alpha(\xi_k) \Delta E_k - \int_{\mathfrak{M}} \alpha(x) dE(x) \right\| < \frac{\varepsilon}{4} \quad (x_{k-1} < \xi_k \leq x_k). \quad (6)$$

Then

$$\left\| \sum_{k=1}^n \Delta E_k A \Delta E_k - \sum_{k=1}^n \Delta E_k \frac{A + A^*}{2} \Delta E_k \right\| < \frac{\varepsilon}{4}. \quad (7)$$

Each segment  $[x_{p-1}, x_p]$  belongs to some segment  $[t_{p'-1}, t_{p'}]$ . Therefore, for  $\lambda \bar{\in} [\alpha(t_{p'-1} + 0) - \varepsilon/4, \alpha(t_{p'}) + \varepsilon/4]$ ,  $\operatorname{Im} \lambda = 0$ ,

$$\left\| \left( A_{E(x_p) - E(x_{p-1})} - \lambda E \right)^{-1} \right\| \leq \left\| \left( A_{E(t_{p'}) - E(t_{p'-1})} - \lambda E \right)^{-1} \right\| < M,$$

$$\left\| \left( A_{E(x_p) - E(x_{p-1})} - \lambda E \right) - \left( \left( \frac{A + A^*}{2} \right)_{E(x_p) - E(x_{p-1})} - \lambda E \right) \right\| < \frac{1}{M} < \left\| \left( A_{E(x_p) - E(x_{p-1})} - \lambda E \right)^{-1} \right\|^{-1}.$$

This proves that the entire spectrum of the operator  $\left( \frac{A + A^*}{2} \right)_{E(x_p) - E(x_{p-1})}$  lies in the segment  $[\alpha(t_{p'-1} + 0) - \varepsilon/4, \alpha(t_{p'}) + \varepsilon/4]$ . Suppose the points  $x_{p-1}$  and  $x_p$  are not endpoints of an interval complementary to  $\mathfrak{M}$ . Then the points  $t_{p'-1}$  and  $t_{p'}$  also have this property, and, consequently,

$$\left\| \Delta E_p \frac{A + A^*}{2} \Delta E_p - \alpha(\xi_p) \Delta E_p \right\| = \left\| \left( \frac{A + A^*}{2} \right)_{\Delta E_p} - \alpha(\xi_p) E \right\| \leq \frac{\varepsilon}{2}. \quad (8)$$

If, however, the points  $x_{p-1}$  and  $x_p$  are endpoints of a complementary interval, then by virtue of (4),  $\Delta E_p \frac{A + A^*}{2} \Delta E_p - \alpha(\xi_p) \Delta E_p = 0$ . Thus inequality (8) holds for all  $p = 1, 2, \dots, n$ , and consequently,

$$\left\| \sum_{k=1}^n \Delta E_k \frac{A + A^*}{2} \Delta E_k - \sum_{k=1}^n \alpha(\xi_k) \Delta E_k \right\| \leq \frac{\varepsilon}{2}. \quad (9)$$

The assertion of the lemma follows from inequalities (7), (9), and (6).

3. Let  $K$  be some bounded operator. To each  $\delta$ -partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the set  $\mathfrak{M}$  we assign the integral sum

$$\sum_{k=1}^n E(\xi_k) K \Delta E_k \quad (x_{k-1} \leq \xi_k \leq x_k, \quad \Delta E = E(x_k) - E(x_{k-1})). \quad (10)$$

If, as  $\delta \rightarrow 0$ , the sums (10) converge in norm to some operator  $B$ , then we shall write

$$B = \int_{\mathfrak{M}} E(x) K dE(x).$$

A completely continuous operator will be called **Volterra** if zero is the only point of its spectrum.

**Theorem\*.** *Let  $A$  be an operator of class  $\Omega$ ; let  $\alpha(x)$  and  $E(x)$  be the functions defined above. Then*

$$A = \int_{\mathfrak{M}} \alpha(x) dE(x) + 2i \int_{\mathfrak{M}} E(x) K dE(x) \quad \left( K = \frac{A - A^*}{2i} \right). \quad (11)$$

**Proof.** If

$$0 = x_0 < x_1 < \dots < x_n = 1$$

is an arbitrary partition of the set  $\mathfrak{M}$ , then

$$A := \sum_{s,t=1}^n \Delta E_s A \Delta E_t = \sum_{k=1}^n \Delta E_k A \Delta E_k + 2i \sum_{s < t} \Delta E_s K \Delta E_t \quad \left( K = \frac{A - A^*}{2i} \right). \quad (12)$$

By virtue of the lemma proved above, there exists such a sequence of partitions of the set  $\mathfrak{M}$  for which the first summand in the sum (12) converges in norm to

$$\int_{\mathfrak{M}} \alpha(x) dE(x).$$

At the same time the second summand will converge in norm to some operator  $B$ . Since the operator

$$2i \sum_{s < t} \Delta E_s K \Delta E_t$$

is completely continuous and some natural power of it is equal to zero, it is Volterra. Consequently, the operator  $B$  is also Volterra. Moreover,

$$2i \sum_{s < t} \Delta E_s K \Delta E_t - \left( 2i \sum_{s < t} \Delta E_s K \Delta E_t \right)^* = 2iK - 2i \sum_{s=1}^n \Delta E_s K \Delta E_s,$$

and therefore

$$\frac{B - B^*}{2i} = K.$$

Let us also note that each subspace  $E(x)H$  is invariant for the operator

$$2i \sum_{s < t} \Delta E_s K \Delta E_t,$$

and hence also for the operator  $B$ . Finally, taking into account that, by (4),

$$(E(b) - E(a))K(E(b) - E(a)) = 0$$

for every interval  $(a, b)$  complementary to  $\mathfrak{M}$ , we obtain

$$B = 2i \int_{\mathfrak{M}} E(x)K dE(x).$$

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\* For the particular case

$$\sum_{\alpha=1}^{\infty} |\omega_{\alpha}|^p < \infty \quad (p > 1),$$

where  $\omega_\alpha$  are the eigenvalues of the operator

$$\frac{A - A^*}{2i},$$

a triangular representation of the form (11) was obtained by I. Ts. Gohberg and M. G. Krein.

*Note: Figure translations are in progress. See original paper for figures.*

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