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Abstract

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FIRST-ORDER SYSTEMS DEFINED ON A RIEMANNIAN MANIFOLD

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In the article ([1]) one special first-order system was considered, containing 8 unknown functions of 4 independent variables. We shall give an n-dimensional analogue of the indicated system, showing that the left-hand side of systems of the type under consideration is written with the aid of the operator of exterior differentiation and of the operator metrically adjoint to it, while the totality of unknowns may be regarded as a totality of differential forms. Such a notation gives the system an invariant character, allowing it to be considered on an arbitrary Riemannian manifold. The orthogonal decompositions that make it possible to elucidate the structure of the system may be obtained in the general case from Kodaira' s theorem ([2]).

For a compact manifold it is most convenient to use decompositions in the form proposed in ([3]). The constructions in ([1]), which are directly generalized to the n-dimensional case, correspond to consideration of a manifold topologically equivalent to a torus. In this case the normalization conditions (1) for the right-hand sides, ensuring solvability, may be interpreted as conditions of orthogonality to all harmonic forms. The conditions on the sought forms ω^p, ensuring the unique determination of the solution, naturally depend on the homological characteristics of the manifold. We shall make some remarks on this below.

Let us note that the operators of the left-hand side of the systems studied, for the case of Euclidean space, were given by a number of authors ([4–6]) in connection with generalizations of integrals of Cauchy type; however, the structure of the operators was not investigated in this connection.

We proceed to the description of the systems (a detailed definition of all concepts used below is contained in ([2])). Let V be an n-dimensional compact Riemannian manifold without boundary. On V one may consider differential forms ω^p of degree p ($p = 0, 1, \dots, n$); moreover, on ω^p there is defined the operator of exterior differentiation d, and for forms of the same degree ω^p, χ^p there is defined the scalar product (ω^p, χ^p). Thus one may consider the operator δ, metrically adjoint to d, i.e., related to it by the relation $(d\omega^p, \chi^{p+1}) = (\omega^p, \delta\chi^{p+1})$. We shall write out in full the systems of equations of interest to us for the case $n \leq 4$. In doing so, for each n we shall have two systems that are "metrically adjoint." For Euclidean space the systems (K), (K^*) are the so-called formally adjoint systems. The forms ω^p are the "unknown" forms; the forms α^k (of degree k) are given.

right-hand sides. The systems have the form

$$\begin{aligned} d\overset{1}{\omega} &= \overset{2}{\alpha}, \\ \delta\overset{1}{\omega} &= \overset{0}{\alpha}; \end{aligned} \qquad (K_2)$$

$$d\overset{0}{\omega} + \delta\overset{2}{\omega} = \overset{1}{\alpha}; \qquad (K_2^*)$$

$$\begin{aligned} d\overset{1}{\omega} + \delta\overset{3}{\omega} &= \overset{2}{\alpha}, \\ \delta\overset{1}{\omega} &= \overset{0}{\alpha}; \end{aligned} \qquad (K_3)$$

$$\begin{aligned} d\overset{0}{\omega} + \delta\overset{2}{\omega} &= \overset{1}{\alpha}, \\ d\overset{2}{\omega} &= \overset{3}{\alpha}; \end{aligned} \qquad (K_3^*)$$

$$\begin{aligned} d\overset{1}{\omega} + \delta\overset{3}{\omega} &= \overset{2}{\alpha}, \\ \delta\overset{1}{\omega} &= \overset{0}{\alpha}, \\ d\overset{3}{\omega} &= \overset{4}{\alpha}; \end{aligned} \qquad (K_4)$$

$$\begin{aligned} d\overset{0}{\omega} + \delta\overset{2}{\omega} &= \overset{1}{\alpha}, \\ d\overset{2}{\omega} + \delta\overset{4}{\omega} &= \overset{3}{\alpha}. \end{aligned} \qquad (K_4^*)$$

The homogeneous system (K_2) is the well-known Beltrami equations. An explicit form of (K_3), (K_4), and the corresponding orthogonal decompositions were considered in ([1]). Let us note that the nature of solvability of (K_2) and (K_2^*) for the sphere, for example, is substantially different: the homogeneous system (K_2) has only the zero solution, whereas for K_2^* we have $\overset{0}{\omega} = c$; $\overset{2}{\omega} = c$ ($\overset{0}{c}, \overset{2}{c}$ are constants). For the torus, also in the system (K_2) the solution is determined by two constants. The way of passing from the system (K) to the system (K^*) is obvious. For example:

$$(d\overset{1}{\omega} + \delta\overset{3}{\omega}, \overset{2}{\chi}) + (\delta\overset{1}{\omega}, \overset{0}{\chi}) + (d\overset{3}{\omega}, \overset{4}{\chi}) = (\overset{1}{\omega}, \delta\overset{2}{\chi} + d\overset{0}{\chi}) + (\overset{3}{\omega}, d\overset{2}{\chi} + \delta\overset{4}{\chi}),$$

and one could have confined oneself to giving the system (K_n). We give, however, the adjoint system also in the general case.

For even n:

$$\begin{aligned}
&d\overset{1}{\omega} + \delta\overset{3}{\omega} = \overset{2}{\alpha}, \\
&d\overset{3}{\omega} + \delta\overset{5}{\omega} = \overset{4}{\alpha}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \\
&d\overset{n-3}{\omega} + \delta\overset{n-1}{\omega} = \overset{n-2}{\alpha}, \\
&\delta\overset{1}{\omega} = \overset{0}{\alpha}, \\
&d\overset{n-1}{\omega} = \overset{n}{\alpha};
\end{aligned} \qquad (K_n)$$

$$\begin{aligned}
&d\overset{0}{\omega} + \delta\overset{2}{\omega} = \overset{1}{\alpha}, \\
&d\overset{2}{\omega} + \delta\overset{4}{\omega} = \overset{3}{\alpha}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \\
&d\overset{n-2}{\omega} + \delta\overset{n}{\omega} = \overset{n-1}{\alpha}.
\end{aligned} \qquad (K_n^*)$$

For odd n:

$$\begin{aligned}
&d\overset{1}{\omega} + \delta\overset{3}{\omega} = \overset{2}{\alpha}, \\
&d\overset{3}{\omega} + \delta\overset{5}{\omega} = \overset{4}{\alpha}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \\
&d\overset{n-2}{\omega} + \delta\overset{n}{\omega} = \overset{n-1}{\alpha}, \\
&\delta\overset{1}{\omega} = \overset{0}{\alpha};
\end{aligned} \qquad (K_n)$$

$$\begin{aligned}
&d\overset{0}{\omega} + \delta\overset{2}{\omega} = \overset{1}{\alpha}, \\
&d\overset{2}{\omega} + \delta\overset{4}{\omega} = \overset{3}{\alpha}, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&d\overset{n-3}{\omega} + \delta\overset{n-1}{\omega} = \overset{n-2}{\alpha}, \\
&d\overset{n-1}{\omega} = \overset{n}{\alpha}.
\end{aligned} \qquad (K_n^*)$$

Transferring the construction of (1) to the system (K_n), we obtain that unique solvability is ensured by conditions of the form

$$\int_V \overset{k}{\alpha}_i \, dV = 0; \qquad \int_V \overset{k}{\omega}_i \, dV = 0; \qquad k = 0, 1, \dots, n; \qquad i = 1, \dots, C_n^k, \tag{1}$$

on the components (regarded as scalar functions—0-forms) of the right-hand sides and of the sought forms.

Let us show how, when these conditions are fulfilled, one can construct a solution, for example, of the system (K_5). Conditions (1) ensure the existence of the decompositions

$$\overset{2}{\alpha} = d\overset{1}{\chi} + \delta\overset{3}{\chi}_{\mathrm{I}}; \qquad \overset{4}{\alpha} = d\overset{3}{\chi}_{\mathrm{II}} + \delta\overset{5}{\chi}.$$

In these decompositions $\overset{5}{\chi}$ is determined uniquely; consequently, the sought form $\overset{5}{\omega}$, satisfying the equation $\delta\overset{5}{\omega} = \delta\overset{5}{\chi}$, is also determined uniquely. Fix arbitrarily $\overset{1}{\chi}, \overset{3}{\chi}_{\mathrm{II}}$ and put

$$\overset{1}{\omega} = \overset{1}{\chi} + d\overset{0}{\chi}; \qquad \overset{3}{\omega} = \overset{3}{\chi}_{\mathrm{II}} + d\overset{2}{\chi},$$

determining $\overset{2}{\chi}, \overset{0}{\chi}$ from the solvable equations

$$\delta d\overset{2}{\chi} = \overset{2}{\alpha} - d\overset{1}{\chi} - \delta\overset{3}{\chi}_{\mathrm{II}}; \qquad \delta d\overset{0}{\chi} = \overset{0}{\alpha} - \delta\overset{1}{\chi}. \tag{2}$$

The constructed $\overset{1}{\omega}, \overset{3}{\omega}, \overset{5}{\omega}$ are determined uniquely and satisfy the system (K_5). The fact that conditions (1) ensure, in the case considered, the unique solvability of the systems (K), (K^*), establishes, in particular, the known fact that the k-th Betti number of the n-dimensional torus coincides with C_n^k.

Let us note in conclusion that when the differentiable manifold admits an indefinite metric tensor, the systems (K), (K^*) may also be associated with this tensor. With a metric tensor of signature equal to one, we obtain, for example, a certain class of hyperbolic systems.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.