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Abstract

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MATHEMATICS

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ON CANONICAL HOMOGENEOUS DIFFERENCE SCHEMES

§ 1. In papers ([1−3]) homogeneous three-point difference schemes were considered for solving a class of boundary-value problems

$$L^{(k,q,f)}u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u + f(x) = 0, \qquad 0 < x < 1,$$

$$u(0) = \mu_1, \qquad u(1) = \mu_2, \tag{1}$$

$$0 < K_1 \leq k(x) \leq K_2, \qquad 0 \leq q(x) \leq K_2, \qquad |f(x)| \leq K_2,$$

depending on the choice of the coefficients $k(x)$, $q(x)$, $f(x)$ from some functional family.

Let $S_N = \{x_0 = 0,\ x_1 = h, \dots,\ x_i = ih, \dots,\ x_N = Nh = 1\}$ be a uniform difference grid. We shall consider homogeneous three-point difference schemes of the form

$$L_h^{(k,q,f)}y_i = L_h^{(k)}y_i - D_i^{(h,q)}y_i + F_i^{(h,f)}, \tag{2}$$

$$L_h^{(k)}y_i = \frac{1}{h^2}\left[B_i^{(h,k)}(y_{i+1} - y_i) - A_i^{(h,k)}(y_i - y_{i-1})\right].$$

Homogeneity of the scheme means that its coefficients have the form

$$A_i^{(h,k)} = A^h[\bar{k}_i(s)], \qquad B_i^{(h,k)} = B^h[\bar{k}_i(s)], \qquad \bar{k}_i(s) = k(x_i{+}sh), \quad -1 < s < 1,$$

$$D_i^{(h,q)} = D^h[q(x_i + sh)], \qquad F_i^{(h,f)} = F^h[f(x_i + sh)], \quad -0.5 < s < 0.5,$$

where A^h, B^h, D^h, and F^h are certain, generally speaking nonlinear, functionals defined on the set of piecewise-continuous functions Q_0 and depending parametrically on the mesh step h.

The initial family of difference schemes is determined by specifying the class of functionals by means of which the coefficients of the scheme are computed.

§ 2. We shall specify the class of functionals $A^h[\psi(s)]$ by means of the following conditions.

(A_1). The functional $A^h[\psi]$ has a third-order differential with respect to h, so that

$$A^h[\psi] = A^{(0)}[\psi] + hA^{(1)}[\psi] + h^2A^{(2)}[\psi] + h^3A^{(3)}[\psi] + h^3\rho(h),$$

where $\rho(h) \to 0$ as $h \to 0$, and the functional $A^{(m)}[\psi]$ has a differential of order $3 - m$ $(m = 0, 1, 2, 3)$, so that, for example, for $m = 1$ one may write

$$A^{(1)}[f + \delta\varphi] = A^{(1)}[f] + \delta A_1^{(1)}[f, \varphi] + \delta^2 A_2^{(1)}[f, \varphi] + \delta^2\rho(\delta),$$

where $\rho(\delta) \to 0$ as $\delta \to 0$.

(A_2) $A^h[\psi]$ and all the functionals $A^{(m)}[\psi]$ $(m = 0, 1, 2, 3)$ are homogeneous functionals of degree 1:

$$A^h[c\psi] = cA^h[\psi], \quad \text{where } c > 0 \text{ is a constant;} \qquad A^{(m)}[c\psi] = cA^{(m)}[\psi],$$

and $A^h[1] = 1$.

(A_3) $A^h[\psi]$ and all the functionals $A^{(m)}[\psi]$ $(m = 0, 1, 2, 3)$ are monotonically nondecreasing, i.e.

$$A^h[\psi_2] \geq A^h[\psi_1], \qquad \text{if } \psi_2 \geq \psi_1.$$

We shall assume that the functionals A^h, B^h, D^h, and F^h satisfy conditions (A_1), (A_2), (A_3), with D^h and F^h linear.

From conditions (A_1) and (A_2) it follows that

$$A^h[k(x+sh)] = k(x)+hk'(x)A_1^{(0)}[s]+h^2\left\{k'(x)A_1^{(1)}[s] + \frac{(k'(x))^2}{k(x)}A_2^{(0)}[s] + \frac{k''(x)}{2}A_1^{(0)}[s^2]\right\}+O(h^3), \tag{3}$$

where $A^{(m)}[\psi(s)] = A^{(m)}[1, \psi(s)]$.

We note that in works ([2], [3]) a narrower class of functionals was studied and, consequently, a narrower initial family of difference schemes.

Item 3. If the functional $A^h[\psi]$ does not depend on the parameter h, then it is called **canonical** and is denoted by $A[\psi]$. A difference scheme $L_h^{(k,q,f)}$, whose coefficients are defined through canonical functionals, is called a **canonical scheme**.

If the condition

$$B_i^{(h,k)} = A_{i+1}^{(h,k)}, \quad \text{i.e.} \quad B^h[\psi(s)] = A^h[\psi(s+1)] \tag{4}$$

is fulfilled for any function from Q_0, then the difference scheme $L_h^{(k)}$ (and $L_h^{(k,q,f)}$) is called **conservative**; it can be written as

$$L_h^{(k)} y_i = \frac{1}{h^2} \Delta \left(A_i^{(h,k)} \nabla y_i \right), \quad \text{where } \Delta y_i = y_{i+1} - y_i, \quad \nabla y_i = y_i - y_{i-1}. \tag{5}$$

From condition (4) it follows that the functional $A^h[\psi(s)]$ does not depend on the values of $\psi(s)$ for $0 < s < 1$, while $B^h[\psi(s)]$ does not depend on the values of $\psi(s)$ for $-1 < s < 0$.

Item 4. One of the characteristics of a difference scheme is its integral order of accuracy with respect to h, i.e., the order of accuracy of the difference $z_i = y_i - u(x_i)$ as $h \to 0$, where $u(x)$ is the solution of problem (1), and y_i is the solution of the difference boundary-value problem

$$L_h^{(k,q,f)} y_i = 0, \quad 0 < i < N, \quad y_0 = \mu_1, \quad y_N = \mu_2. \tag{6}$$

The function z_i is determined by the conditions

$$L_h^{(k,q)} z_i = -\varphi_i, \quad z_0 = 0, \quad z_N = 0, \tag{7}$$

where $\varphi_i = \varphi(x_i, h; u(x_i))$.

Recall that the difference $\varphi(x, h; v) = L_h^{(k,q,f)} v - L^{(k,q,f)} v$, where $v = v(x)$ is any sufficiently smooth function, is called the **approximation error of the scheme**.

If $v = u(x)$ is the solution of the differential equation (1), then we shall speak of the approximation error $\varphi(x, h; u)$ on the solution of the differential equation. It may turn out, generally speaking, that the order of the approximation error of the scheme on the solution is higher than the order of approximation on the class of smooth functions $v(x)$.

In the present paper we study the order of accuracy of difference schemes of the initial family in the class C_m of functions having a continuous derivative of order m ([1]).

The order of approximation of the scheme $L_h^{(k,q,f)}$ is determined by the values of the moments of the functionals A^h, B^h, D^h, and F^h.

A necessary and sufficient condition for first-order approximation of the scheme $L_h^{(k,q,f)}$ has the form

$$B_1^{(0)}[s] - A_1^{(0)}[s] = 1. \tag{8}$$

In order that the scheme have second-order approximation ($\varphi(x, h; v) = O(h^2)$), it is necessary and sufficient that the conditions

$$B_1^{(0)}[s] = -A_1^{(0)}[s] = 0.5, \quad B_1^{(1)}[s] = A_1^{(1)}[s]; \tag{9}$$

$$B_2^{(0)}[s] = A_2^{(0)}[s], \quad B_1^{(0)}[s^2] = A_1^{(0)}[s^2], \quad D^{(0)}[s] = F^{(0)}[s] = 0. \tag{10}$$

The conditions (10) for a symmetric scheme are satisfied automatically. Here an essential role is played by normalization conditions, for example $A^h[1] = 1$, from which it follows that $A^{(0)}[1] = 1$, $A^{(m)}[1] = 0$ for $m > 0$.

Item 5. Theorem 1. *In order that the original scheme $L_h^{(k,q,f)}$ have n-th ($n = 1, 2$) integral order of accuracy in $C_{m_k,m_q,m_f} = \{C_{m_k}, C_{m_q}, C_{m_f}\}$, $m_k \geq n+1$, $m_q \geq n$, $m_f \geq n$, it is necessary and sufficient that it have n-th order of approximation.*

Theorem 2. *Every conservative scheme from the original class has first integral order of accuracy in C_{m_k,m_q,m_f} for $m_k \geq 2$, $m_q \geq 1$, $m_f \geq 1$.*

Theorem 3. *Every canonical symmetric conservative scheme has second order of accuracy in C_{m_k,m_q,m_f} for $m_k \geq 3$, $m_q \geq 2$, $m_f \geq 2$.*

Item 6. Along with studying the accuracy of difference schemes for solving a certain class of problems, one must be able to estimate the error permitted when solving, by a given scheme, each individual problem from the class under consideration. Such an accuracy estimate can be achieved by studying the asymptotics of the solution of the difference boundary-value problem as $h \to 0$. First of all one must find expansions in h (asymptotics) of the approximation error $\varphi(x, h; v) = L_{hv} - Lv$.

Here we restrict ourselves to computing the coefficient of the lowest power of h for a canonical conservative scheme. Introducing $p(x) = 1/k(x)$, instead of $L_h^{(k)}$ we shall write

$$L_h^{(p)} y_i = \frac{1}{h^2} \Delta \left(\frac{\nabla y_i}{A_i} \right), \qquad A_i = A[p(x_i + sh)],$$

where $A[\psi]$ is the canonical functional. If $L_h^{(p,q,f)}$, moreover, is a symmetric scheme, then

$$\varphi(x, h; v) = h^2 \Phi(x, v) + O(h^4), \qquad \Phi(x, v) = \frac{1}{12} \left\{ \left[\frac{(pL^{(p)} v)'}{p} \right]' \right.$$

$$\left. - 12 A_2[s] \left[\frac{(p')^2 v'}{p^3} \right]' - 6 \left[\frac{p'' v'}{p^2} \right]' \left(A_1[s^2] - \frac{1}{3} \right) + 6 \left(F[s^2] f'' - D[s^2] q'' v \right) \right\},$$

$$L^{(p)} v = \left(\frac{v'}{p} \right)'.$$

In the case of the best canonical scheme (see (3))

$$A[\psi] = \int_{-1}^{0} \psi(s)\, ds,$$

$$D[\psi] = F[\psi] = \int_{-0.5}^{0.5} \psi(s)\, ds$$

we have

$$A_1[s^2] = \frac{1}{3}, \qquad A_2[s] = 0, \qquad D[s^2] = F[s^2] = \frac{1}{12},$$

and the expression for Φ is greatly simplified.

We shall say that difference schemes $L_h^{(p,q,f)}$ and $\overline{L}_h^{(p,q,f)}$ are **asymptotically equivalent in the sense of approximation** if $\Phi(x, v) = \overline{\Phi}(x, v)$ and, consequently,

$$\varphi(x, h; v) - \overline{\varphi}(x, h; v) = O(h^4).$$

In our case this means that

$$A_2[s] = \overline{A}_2[s], \qquad A_1[s^2] = \overline{A}_1[s^2], \qquad D[s^2] = \overline{D}[s^2], \qquad F[s^2] = \overline{F}[s^2].$$

In particular, the scheme $\overline{L}_h^{(p)}$, for which

$$\overline{A}_i = \frac{1}{3}\left(p_{i-1} + p_{i-0.5} + p_i\right),$$

is asymptotically equivalent to the best canonical scheme $L_h^{(p)}$, for which

$$A_i = \int_{-1}^{0} p(x_i + sh)\, ds.$$

* The differentiability conditions on the coefficients $k(x)$, $q(x)$, and $f(x)$ can be substantially weakened. This

p. 7. Using the asymptotics for $\varphi(x, h; v)$, it is not difficult to obtain the asymptotic expansion in h of the solution of the difference boundary-value problem (6) in the form $y_i = u(x_i) + h^2 \tilde{z}(x_i) + O(h^4)$, where $\tilde{z}(x)$ is the function determined from the conditions $L^{(p,q)}\tilde{z} = -\Phi(x, u)$, $\tilde{z}(0) = \tilde{z}(1) = 0$.

p. 8. The study of the asymptotics of the approximation error makes it possible to indicate a method for constructing difference schemes of increased accuracy. Considering, for a scheme L_h corresponding to the linear differential operator L, the asymptotics

$$\varphi(x, h; u) = h^n \Phi u + \cdots$$

and replacing the differential operator Φu by the difference operator $\Phi_h u$, we obtain the difference scheme

$$\tilde{L}_h u = L_h u - h^n \Phi_h u,$$

which has a higher order of approximation (higher than n) on the solution $u = u(x)$ of the equation $Lu = 0$ as compared with the scheme L_h. It is expedient first to transform the operator Φu to an operator of lower order, using for this purpose the equation $Lu = 0$. In this case one can ensure that the operator $\tilde{L}_h$ will have the same domain of definition as the operator L_h.

In our case, application of this method makes it possible to construct various three-point difference schemes $\tilde{L}_h^{(p,q,f)}$ of increased order of accuracy. Thus, for example, if $L_h^{(p,q,f)}$ is the best canonical scheme or a scheme asymptotically equivalent to it, then the three-point homogeneous scheme

$$\begin{aligned}\tilde{L}_h^{(p,q,f)} y_i &= L_h^{(p)}\left(1 - \frac{h^2}{12} p_{iq} i\right) y_i - \tilde{D}_i y_i + \tilde{F}_i \\ &= L_h^{(p,q,f)} y_i - \frac{h^2}{12}\left[L_h^{(p)}(p_{iq} i y_i - p_i f_i) + \frac{1}{2}\left(L_h^{(1)} f_i - y_{iL} h^{(1)} q_i\right)\right]\end{aligned}$$

has 4th integral order of accuracy in the class of smooth coefficients.

p. 9. Suppose that at $x = 0$ a boundary condition of the 3rd kind is prescribed,

$$lu = \frac{u'(0)}{p(0)} - \sigma u(0) = \mu_1.$$

For the corresponding difference boundary operator l, one can consider the approximation error both on any sufficiently smooth function $v = v(x)$ and on the solution of the differential equation $L^{(p,q,f)}u = 0$. By analogy with p. 8 one can construct a two-point difference boundary operator $\tilde{l}_h$ of arbitrary order of approximation on the solution $u = u(x)$. For example, the operator

$$l_{hy} = \frac{1}{hA_1}(y_1 - y_0) - y_0(\sigma + 0.5hq_0) + 0.5hf_0, \qquad A_1 = A[p(x_1 + sh)]$$

has 2nd order of approximation on the solution ($\tilde{l}_{hu} - lu = O(h^2)$) (cf. with ([4])).

It can be shown that the solution of the difference boundary-value problem

$$\tilde{L}_h^{(p,q,f)} y_i = 0, \qquad \tilde{l}_{hy} i = \mu_1, \qquad y_N = \mu_2$$

has n-th order of accuracy, i.e. $y_i - u(x_i) = O(h^n)$, if the operators $\tilde{l}_h$ and $\tilde{L}_h^{(p,q,f)}$ have n-th order of approximation on the solution $u = u(x)$ of the equation $L^{(p,q,f)}u = 0$.

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Note: Figure translations are in progress. See original paper for figures.