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Abstract

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MATHEMATICS

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On the Composition of Singular Kernels

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Let $x, y, t, x', y', \ldots$ be points of the $(m+1)$-dimensional Euclidean space E_{m+1}. Denote by $r(x, y)$ the distance between the points x and y, and by $\rho_x(y, t)$ the expression

$$\rho_x(y,t) = \min\{r(x,y),\, r(x,t)\}.$$

Constants that do not depend on the basic variables will be denoted by the same letter c.

Suppose that in E_{m+1} there is given a closed m-dimensional Lyapunov surface S, and on (S, S) functions $K_1(x, y)$ and $K_2(x, y)$ are defined, continuous on (S, S), except, possibly, at points of the type (x, x), and satisfying the conditions:

$$1^\circ.\quad |K_1(x,y)| \le \xi_1(r(x,y)),\quad |K_2(x,y)| \le \xi_2(r(x,y)).$$
$$2^\circ.\quad |K_1(x,y') - K_1(x,y'')| \le \zeta_1(r(y',y''))\mu_1(\rho_x(y',y'')).$$
$$3^\circ.\quad |K_2(x',y) - K_2(x'',y)| \le \zeta_2(r(x',x''))\mu_2(\rho_y(x',x'')).$$

Here $\xi_1(\tau)$, $\xi_2(\tau)$, $\mu_1(\tau)$, $\mu_2(\tau)$ are decreasing, while $\zeta_1(\tau)$ and $\zeta_2(\tau)$ are increasing functions on $(0, \infty)$. Suppose, further, that there exist matrices $\|A^{(1)}_{i,j}(x)\|$ and $\|A^{(2)}_{i,j}(x)\|$, continuous on S, such that the quadratic forms

$$\sum_{i,j=1}^{m+1} A^{(1)}_{i,j}(x)\tau_i\tau_j, \qquad \sum_{i,j=1}^{m+1} A^{(2)}_{i,j}(x)\tau_i\tau_j$$

are positive definite on S, and for every $x \in S$ the limits exist:

$$4^\circ.\quad \lim_{\varepsilon\to 0}\int_{S_1(\varepsilon;x)} K_1(x,y)\,dS_y = \int_S K_1(x,y)\,dS_y.$$

$$5^\circ. \quad \lim_{\varepsilon\to 0}\int_{S_2(\varepsilon;x)} K_2(y,x)\,dS_y = \int_S K_2(y,x)\,dS_y.$$

Here dS_y is the element of area of the surface S at the point y; $S_k(\varepsilon;x) = S - \bar{S}_k(x;\varepsilon)$ $(k=1,2)$; $S_k(x;\varepsilon)$ is the set of those points $y \in S$ for which $\sigma_k(x,y) < \varepsilon$;

$$\sigma_k^2(x,y) = \sum_{i,j=1}^{m+1} A_{i,j}^{(k)}(x)(x_i - y_i)(x_j - y_j).$$

We now define the composition of the kernels $K_1(x,y)$ and $K_2(x,y)$ by the formula

$$K(x,y) = \int_S K_1(x,t)K_2(t,y)\,dS_t, \tag{1}$$

where

$$\int_S K_1(x,t)K_2(t,y)\,dS_t = \lim_{\varepsilon,\delta\to 0}\int_{S(\varepsilon,\delta)} K_1(x,t)K_2(t,y)\,dS_t,$$

$$S(\varepsilon,\delta) = S_1(\varepsilon;x)\cap S_2(\delta;y).$$

Theorem 1. If $K_1(x,y)$, $K_2(x,y)$ satisfy conditions 1°–5°, then the composition of these kernels $K(x,y)$, defined by formula (1), is continuous on (S,S), except possibly at points of the type (x,x), and near these points admits the estimate

$$c|K(x,y)| \le \mu_1(r)\int_0^r \tau^{m-1}\xi_2(\tau)\zeta_1(\tau)\,d\tau + \mu_2(r)\int_0^r \tau^{m-1}\xi_1(\tau)\zeta_2(\tau)\,d\tau + \int_r^c \tau^{m-1}\xi_1(\tau)\xi_2(\tau)\,d\tau + \xi_1(r)\eta_2(r) + \xi_2(r \tag{2}$$

where $r = cr(x,y)$; $\eta_1(\tau)$, $\eta_2(\tau)$ are increasing functions on $(0,\infty)$, chosen so that

$$\left|\int_{S_1(x;\varepsilon)} K_1(x,y)\,dS_y\right| \le \eta_1(\varepsilon), \qquad \left|\int_{S_2(x;\varepsilon)} K_2(y,x)\,dS_y\right| \le \eta_2(\varepsilon).$$

It follows from Theorem 1 that if the kernel $K_2(x,y)$ is quasisingular, i.e. if

$$\int_0^c \tau^{m-1}\xi_1(\tau)\,d\tau < \infty,$$

then restrictions 3° and 4° become superfluous, and inequality (2) takes the form

$$c|K(x,y)| \leq \xi_2(r)\int_0^r \tau^{m-1}\xi_1(\tau)\,d\tau + \mu_1(r)\int_0^r \tau^{m-1}\zeta_1(\tau)\xi_2(\tau)\,d\tau + \int_r^c \tau^{m-1}\xi_1(\tau)\xi_2(\tau)\,d\tau + \xi_1(r)\eta_2(r). \tag{3}$$

If $K_2(x,y)$ is quasisingular, then an analogous inequality is valid. If both kernels are quasisingular, then restrictions 2°—5° become superfluous and

$$c|K(x,y)| \leq \xi_1(r)\int_0^r \tau^{m-1}\xi_2(\tau)\,d\tau + \xi_2(r)\int_0^r \tau^{m-1}\xi_1(\tau)\,d\tau + \int_r^c \tau^{m-1}\xi_1(\tau)\xi_2(\tau)\,d\tau. \tag{4}$$

Let $K_1(x,y)$ satisfy two further additional conditions:

6°.

$$|K_1(x',y) - K_1(x'',y)| \leq \xi_3(r(x',x''))\mu_3(\rho_y(x',x'')).$$

7°.

$$\left|\int_{S_1(x;\varepsilon)} [K_1(x,t) - K_1(y,t)]\,dS_t\right| \leq \nu_1(r(x,y))\nu_2(\varepsilon).$$

Here $\xi_3(\tau)$ and $\nu_1(\tau)$ are increasing, $\mu_3(\tau)$ and $\nu_2(\tau)$ are decreasing functions on $(0,\infty)$, and $r(x,y) \leq c\varepsilon$.

Condition 7° follows from the preceding ones if $K_1(x,y)$ is representable in the form

$$K_1(x,t) = K_1'(x,t) + K_1''(x,t), \tag{5}$$

where $K_1''(x,t)$ is quasisingular, and $K_1'(x,t)$ is a purely singular kernel, i.e.

$$\int_{S_1(x;\varepsilon)} K_1(x,y)\,dS_y = 0$$

for every $x \in S$ and $\varepsilon \in (0,c)$. The kernels considered in papers (1^{-3}) admit the representation (5).

Theorem 2. *If $K_1(x,y)$, $K_2(x,y)$ satisfy conditions 1°–7°, then for $\rho \geq r$*

$$c|K(x',y) - K(x'',y)| \leq v_1(r)v_2(\rho)\xi_1(\rho) + \zeta_2(r)\eta_1(r)\mu_2(\rho) +$$

$$+\zeta_3(r)\eta_2(r)\mu_3(\rho) + \mu_2(\rho)\int_0^r \tau^{m-1}\xi_1(\tau)\zeta_2(\tau)\,d\tau +$$

$$+\zeta_3(r)\mu_2(\rho)\int_r^\rho \tau^{m-1}\mu_3(\tau)\zeta_2(\tau)\,d\tau+\mu_1(\rho)\int_0^r \tau^{m-1}\zeta_1(\tau)\xi_2(\tau)\,d\tau+$$

$$+\zeta_3(r)\mu_3(\rho)\int_0^\rho \tau^{m-1}\xi_2(\tau)\,d\tau+\zeta_3(r)\int_\rho^c \tau^{m-1}\mu_3(\tau)\xi_2(\tau)\,d\tau,$$

and for $\rho < r$

$$c|K(x',y)-K(x'',y)|\le \xi_2(\rho)\eta_1(r)+\xi_1(\rho)\eta_2(r)+$$

$$+\mu_2(\rho)\int_0^r \tau^{m-1}\xi_1(\tau)\zeta_2(\tau)\,d\tau+\mu_1(\rho)\int_0^r \tau^{m-1}\xi_2(\tau)\zeta_1(\tau)\,d\tau+$$

$$+\zeta_3(r)\int_\rho^c \tau^{m-1}\mu_3(\tau)\xi_2(\tau)\,d\tau+\int_\rho^r \tau^{m-1}\xi_1(\tau)\xi_2(\tau)\,d\tau,$$

where $\rho = c\rho_y(x',x'')$.

Let us note that this inequality is substantially simplified if K_1, or K_2, or both of these kernels are quasisingular.

An analogous estimate and remark are valid for the difference $|K(x,y')-K(x,y'')|$.

The two theorems stated play the main role in the study of integral equations containing multidimensional singular integrals distributed on Lyapunov manifolds, as well as in the study of systems of such equations. With the help of these theorems one can easily obtain formulas for interchanging the order of integration in multidimensional singular integrals of the indicated type and thereby refine and generalize the results of Giraud (^{2}).

Let us now consider kernels of special types. If a function $K(x,y)$, defined and continuous on (S,S), except possibly at points of the type (x,x), admits the estimate

$$|K(x,y)|\le cr^{\alpha-m}(x,y)\lg^p cr^{-1}(x,y),$$

then we shall write $K\in N(\alpha,p)$. By $N(\alpha,p)$, when $\alpha=m$ and $p\le 0$ or $\alpha>m$ and p is an arbitrary real number, is meant the class of bounded functions $K(x,y)$. We shall also denote $N(0,0)$ by N. If $K(x,y)\in N(\alpha,p)$ and satisfies the conditions

$$|K(x',y)-K(x'',y)|\le$$

$$\le c[\rho_y(x',x'')]^{\alpha-m-\beta_1}[\lg c\rho_y^{-1}(x',x'')]^{p-q_1} r^{\beta_1}(x'',y)\lg^{q_1} cr^{-1}(x'',y), \tag{6}$$

$$|K(x,y')-K(x,y'')|\le$$

$$\le c[\rho_x(y',y'')]^{\alpha-m-\beta_2}[\lg c\rho_y^{-1}(y',y'')]^{p-q_2} r^{\beta_2}(y'',y)\lg^{q_2} cr^{-1}(y'',y),$$

then we shall write $K \in L(\alpha, p; \beta_1, q_1; \beta_2, q_2)$. If $K \in N(\alpha, p)$ and satisfies only condition (6), then we shall write $K \in L_1(\alpha, p; \beta_1, q_1)$. The notation $K \in L_2(\alpha, \beta; \beta_2, q_2)$ is understood analogously. If $K \in N(\alpha, p)$ for every real p, then we shall write $K \in N(\alpha, \infty)$. The notations $K \in L(\alpha, \infty; \beta_1, p_1; \beta_2, p_2)$, $L(\alpha, \infty; \beta_1, \infty; \beta_2, p_2)$, ... have an analogous meaning.

From inequality (4), with the aid of certain estimates indicated in (4), one can show that if $K_1 \in N(\alpha, p)$ and $K_2 \in N(\beta, q)$, then, for arbitrary real p and q, $\alpha, \beta > 0$, $\alpha + \beta < m$, $K \in N(\alpha+\beta, p+q)$; for $\alpha, \beta > 0$, $\alpha + \beta = m$, $p + q + 1 > 0$, $K \in N(m, p+q+1)$; K is bounded for $\alpha + \beta > 0$, $\alpha + \beta = m$ and $p + q + 1 < 0$, or for arbitrary real p, q, $\alpha > 0$, $\beta > 0$, and $\alpha + \beta > m$. If $K_1 \in N(\alpha, \infty)$ and $K_2 \in N(\beta, \infty)$, then $K \in N(\alpha + \beta, \infty)$ for $\alpha, \beta > 0$, $\alpha + \beta < m$, and K is bounded for $\alpha + \beta \ge m$.

These propositions somewhat generalize the well-known theorem on the composition of quasisingular kernels. With the aid of these generalizations and the results obtained in papers (3,5,6), one can supplement note (7), taking as the weight function the expression

$$\rho(y) = \prod_k r^{\alpha_k}(y, x_k)\lg^{p_k} cr^{-1}(y, x_k). \tag{7}$$

For $m = 1$ this makes it possible to extend the results of paper (8) to the case of weight functions of the form (7).

From inequality (3), with the aid of the estimates from (4) mentioned above, one can show that if $K_1 \in L_2(\alpha, p; \beta, q)$, and $K_2 \in N$ and satisfies condition 5°, then, for arbitrary real p and q, $\alpha, \beta > 0$, $\alpha < m$, the composition K belongs to the class $N(\alpha, p)$; for arbitrary real p, $\beta = 0$, $0 < \alpha < m$, $q < -1$, or for arbitrary real q, $\alpha = 0$, $\beta > 0$, $p < -1$, or for $\alpha = \beta = 0$, $p, q < -1$, the composition K belongs to the class $N(\alpha, p+1)$. If $K_1 \in L_2(\alpha, \infty; \beta, q)$, and $K_2 \in N$ and satisfies condition 5°, then for arbitrary real q and $\alpha, \beta > 0$, or for $q < -1$, $\beta = 0$ and $\alpha \ge 0$, the composition K belongs to the class $N(\alpha, \infty)$.

These propositions generalize some results of Giraud (2) and show that the composition of a quasisingular and a singular kernel yields a quasisingular kernel.

Analogous consequences are also obtained from other inequalities; with their help one can study the composition of kernels of the classes indicated above.

If, for example, $K_2(x,y)$ does not depend on y, then from Theorem 2 one can obtain a generalization of the results of papers ([9],[10]). Further, if $\xi_2(\tau) = c$, $\mu_2(\tau) = c$, then from the same theorem one obtains a corollary which is the extension to multidimensional singular integrals of one result of the monograph ([11]) (see § 20) and its generalization.

We also note that the kernels encountered in the study of boundary-value problems for systems of elliptic partial differential equations satisfy the requirements set forth above.

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Note: Figure translations are in progress. See original paper for figures.

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