MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND and M. L. TSETLIN

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND and M. L. TSETLIN

ON CONTINUAL MODELS OF CONTROL SYSTEMS

A number of works have been devoted to the construction of mathematical models of control systems that imitate the functions of the nervous system. These works, and the theory of automata (logical networks) that arose in connection with them, have proved useful for studying the principles of construction and functioning of the logical circuits of computing machines. However, the discreteness in elements and in time that underlies them makes them rather ineffective for describing the functioning of systems of any complexity of the biological type. It seems reasonable to us, instead of considering a large number of individual elements with a complex structure of connections between them, to consider continual models*. Representations of this kind were used in the important and interesting work of N. Wiener and A. Rosenblueth ([1]) to describe the mechanism of fibrillation of the cardiac muscle. We shall construct such a continuous medium phenomenologically, considering at the same time various "natural" assumptions about its properties. Let us note that in a physiological experiment as well, the isolation of an individual element is a difficult and not always sufficiently meaningful task, so that a description of the medium not through separate elements but directly seems appropriate. Here we shall describe the simplest model with properties 1°–3°. In this sense the note is preliminary, and we hope in the future to construct continual models that better approximate physiological prototypes.

We shall call an **active tissue** a medium possessing the following properties**:

1°. Each point of the medium is capable of excitation, which is instantaneous. During the time R after the moment of excitation, the point cannot be excited. The quantity R is called the **refractory time**.

We shall call the **phase** $\tau(x,t)$ of a point x at the moment t the time elapsed since the last excitation of this point. For example, if the point x was last excited at the moment $t=0$, then $\tau(x,t)=t$. If $\tau(x,t)<R$, then we shall say that the point is in the **refractory phase**.

2°. Excitation can propagate in the medium. The speed $c(x,t)$ of propagation of excitation at the point x at the moment t depends on the phase of this point, $c(x,t)=\varphi[\tau(x,t)]$. The function $\varphi(\tau)$ is defined for all $\tau \geq R$. Propagation of

Fig. 1

Figure 1: Fig. 1

excitation is impossible through regions that are in the refractory phase. Thus, the state of each point of the system is determined by whether it is excited at the given moment or not, and by its phase τ.

---

* In this note we shall confine ourselves to continual models in the simplest sense of the word; namely, in a continuous medium we shall understand nearness as geometrical nearness. In general, however, such nearness may be understood as nearness in "phase space." (For example, we may understand the distance between points A and B as the time during which a signal from A arrives at B.)

** Properties 1°, 2° were, in essence, already considered by Wiener and Rosenblueth in ([1]). These properties, together with 3°, are convenient not only for considering phenomena in the myocardium, but also for constructing control systems.

The propagation of excitation is regarded as the propagation of a "discontinuity" of the state of the system, whose front moves with velocity c along the normal to the "discontinuity." One could also consider a model in which the excitation front has a finite extent. The propagation of excitation can then be compared with the propagation of flame in a combustible medium.*

3°. A point possesses **spontaneous activity**. This means that after a time T following the last excitation the point is again excited spontaneously (provided, of course, that before this it has not been excited under the action of its neighbors). The quantity T is called the **period of spontaneous activity**.

Let us consider some examples of processes that can occur in such a tissue.

1. Consider a mode of operation called a ring rhythm. There is a thin filament of length l, closed into a ring. Suppose that spontaneous activity is absent and that the velocity of propagation of excitation c is constant. As initial conditions take the distribution of phases $\tau(x, 0)$ indicated in Fig. 1 by the solid line. Thus, at the initial instant the point θ_0 is excited, and its neighbors on the right (the segment $\theta_0\theta'$) are in the refractory phase. Therefore the propagation of excitation will proceed to the left from the point θ_0, while the points situated to the right of θ_0 will gradually leave the refractory phase (their phases will increase). By the time t the distribution of phases $\tau(x, t)$ will have the form shown in Fig. 1 by the dashed line. The length of the segment $\theta_0\theta_t$ is, obviously, ct. Since the beginning and the end of the segment are identified, the excited point, after the time $t = l/c$, will return to the position θ_0, and the distribution of phases will coincide with the initial one.

Fig. 1

In this example one can see the role of refractoriness, which ensures the possibility of one-sided propagation of excitation. In this sense refractoriness, which gives point A the possibility of acting on point B, while point B no longer acts on point A, is opposite to Huygens' principle.** Below we shall return again to modes of this type.

2. Now consider an active tissue, and let T be the period of its spontaneous activity. If all points of the tissue were excited simultaneously at the instant $t = 0$, then subsequently they will be excited simultaneously at the instants $t = T, 2T, \ldots$. Thus, on the phase graph the distribution of phases at the initial instant coincides with the axis of abscissas, and then rises upward with velocity equal to unity. Upon reaching the instant $t = T$, this straight line again falls to the axis. Since, however, the phases $0, T, 2T$ are indistinguishable, it will be more convenient for us to depict the phase distribution as continuously rising, and by phase to understand the distance from the corresponding point of the phase-distribution curve to the straight line $t = nT$, $nT \leq \tau(x,t) < (n+1)T$.

We shall say that interaction is absent in the system if any point of the tissue is excited spontaneously earlier than it could be excited under the action of excited neighboring points. In the absence of interaction, the phase-distribution curve moves upward parallel to itself. It is not difficult to verify that interaction will be absent for all initial distributions for which the slope of the tangent is less than c^{-1}. For a system of a larger number of dimensions, the condition for absence of interaction takes the form: $\operatorname{grad} \tau(x,t) \leq c^{-1}$, $x = (x^1, x^2, x^3)$ are the coordinates of a point of the tissue. We shall see that when this

* A model of flame propagation would probably be useful in studying the propagation of excitation along an axon or a muscle fiber. A model could be provided by the equation (2) $\partial^2 u/\partial t^2 = A\, \partial^2 u/\partial x^2 + F(u)$, or by systems of equations of this type.

** An important feature of Huygens' principle is its reciprocity.

conditions; in particular, in the presence of ruptures, after some time there will again be attained a distribution of phases satisfying this condition.

The medium described possesses, in a certain sense, **reliability**. Indeed, if for some collection of points of the medium $x^1, \ldots, x^n$ the periods of spontaneous activity were randomly increased, this would not lead to a noticeable change in the phase distribution, since these points will be excited under the influence of their neighbors with an infinitely small delay. On the other hand, a random decrease of the period of spontaneous activity by an amount ΔT will lead to a change of the phase distribution only in a region of radius $c\Delta T$. In this respect the active tissue we are considering differs from discrete models of logical networks, in which a change in the properties of an individual element distorts the operation of the entire network. Let us now show that such active tissue can perform memory functions. For simplicity, consider the initial phase distribution $\tau(x,0)=0$, and suppose that at the time $t=R+\varepsilon$ the point x_0 is excited. The phase of the point x_0 at the time $t=R+\varepsilon$, as we have already agreed, will be represented as equal to T (curve 1 in Fig. 2). In the same figure the changes of the phase distribution at various instants of time are shown (curves 2, 3, 4). It is not difficult to see that, beginning with the moment $t=T$, the curve of the phase distribution ceases to change. Thus, excitation of the point x_0 has entailed a persistent change of the phase distribution in the cone (characteristic) with base radius $c(T-R-\varepsilon)$, so that the place of excitation and its phase have been "remembered."

Fig. 2

Figure 3 shows how the phase distribution is established if, with the initial distribution $\tau(x,0)=0$, the point x_0 is excited successively at the moments $t=R+\varepsilon$ and $t=2R+2\varepsilon$. The radius of the base of the characteristic cone is then equal to $2c(T-R-\varepsilon)$. This example shows that each point of the medium can serve as a counter of the number of elementary excitations. We also note that in this case, in the established regime, there are excited points at any instant of time, whereas interaction is absent *.

Fig. 3

3. Let us now consider a thin segment of tissue of length l, devoid of spontaneous activity; the velocity $c(\tau(x,t))$ of propagation of excitation in it is specified for values $\tau \geq R$ and increases monotonically with increasing τ. Suppose now that the point $x=0$ is periodically excited with pe-

---

* One could introduce a measure of interaction in the system in the following way. For a point x excited at a moment lying between t and $t+\Delta t$, such a measure would be the quantity $T-\tau(x,t)$. A measure of interaction for the whole system at the moment t could be taken to be the expression

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Sigma}(T-\tau(x,t))\,d\sigma,$$

where Σ is the set of points excited during the time from t to $t+\Delta t$. In example 2 the interaction tends to zero as t increases. It is possible, however, to construct examples (see, for instance, (1)) in which the interaction tends to a constant value as $t\to\infty$. Regimes in which the interaction tends to zero we shall call normal, and the remaining ones special. (Special regimes arising in the heart would naturally be called fibrillation regimes.)

with period T, and consider the process of propagation of excitation impulses. Denote by $y_k(x)$, $k=1,2,...$, the time of passage of the k-th impulse from point

0 to point x. Then the process of passage of the impulses will be described by the sequence of differential equations

$$y_k'(x) = c^{-1}(T + y_k - y_{k-1})$$

with boundary conditions $y_k(0) = 0$ and initial data $y_0(x) = \varphi(x)$.

It can be shown that for an arbitrary function $\varphi(x)$ the formula

$$\lim_{k\to\infty} y_k(x) = xc^{-1}(T)$$

holds, i.e., the propagation of excitation under periodic action in the limit occurs with constant velocity. This consideration also indicates a method for the experimental determination of the function $c(\tau)$, which thus reduces to measuring the established time interval between the excitation of the beginning and the end of the fiber.

In an analogous way one may also consider the processes of propagation of excitation in a ring. Let $\tau(x, 0) = \varphi(x)$, and let an impulse begin to propagate from the point $x = 0$ in some direction. Further let $z_k(x)$ be the time spent by the k-th impulse in passing from point 0 to point x. Then the process will be described by the sequence of differential equations

$$z_k'(x) = c^{-1}(z_{k-1}(l) - z_{k-1}(x) + z_k(x)), \quad k = 1, 2, \dots;$$

$$z_0(x) - z_0(l) = \varphi(x).$$

In this case, too, it can be shown that for an arbitrary function $\varphi(x)$

$$\lim_{k\to\infty} z_k(x) = xt_0 l^{-1},$$

where t_0 can be found from the equality

$$t_0 = lc^{-1}(t_0) \quad (\text{for } l \geq RC(R)).$$

Analogous results are also obtained in the study of the circulation of groups of impulses around a ring. Discrete electronic models of this type were studied in work (3).

We express our gratitude to the participants of the seminar—to the physiologists I. S. Balakhovsky, V. S. Gurfinkel, V. B. Malkin, and M. L. Shik—for their valuable participation in the discussion of the questions presented here.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
13 I 1960

CITED LITERATURE

1. N. Wiener, A. Rosenblueth, *Arch. del Inst. de Cardiologia de Mexico*, **16**, 3-4, 205 (1946).

2. I. M. Gelfand, *UMN*, **14**, 2(86) (1959).

3. A. F. Ivanov, V. R. Telesnin, *Izv. Vyssh. uchebn. zav.*, Radiophysics, **2**, No. 1 (1959).

Note: Figure translations are in progress. See original paper for figures.