ON THE DISTRIBUTION OF SINGULARITIES OF ONE CLASS OF FUNCTIONS

The theorem on the distribution of singularities of one class of Dirichlet series of the form

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Abstract

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MATHEMATICS

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ON THE DISTRIBUTION OF SINGULARITIES OF ONE CLASS OF FUNCTIONS

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The theorem on the distribution of singularities of one class of Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s), \qquad 0 \leq \lambda_1 < \lambda_2 < \cdots, \tag{1}$$

with the condition on the exponents

$$\lim(\lambda_{n+1} - \lambda_n) \geq h > 0, \tag{2}$$

proved by Agmon ([3]), is generalized in the present paper to a broader class of functions. In addition, a certain refinement of this theorem is given.

Definition. We shall say that a function $f(s)$ belongs to the class $B(r, h)$ if $f(s)$ is representable for $\sigma > 0$ ($s = \sigma + it$) as the limit (uniformly convergent in any bounded closed domain belonging to the half-plane $\sigma > 0$) of a sequence of "Dirichlet polynomials" :

$$D_k(s) = \sum_{n=1}^{n_k} \sum_{j=0}^{\alpha_n - 1} a_{n,k,j}\, s^j \exp(-\gamma_n s), \tag{3}$$

where the sequence of exponents $\{\lambda_n\}$, in which each γ_n occurs α_n times, satisfies the condition

$$\lim(\lambda_{n+r} - \lambda_n) \geq h > 0. \tag{4}$$

We shall denote by $[e^{-xs}, \delta_0, \delta_1, \dots, \delta_n]$ the finite divided difference of the function e^{-xs} at the points $\delta_0, \delta_1, \dots, \delta_n$. From the results of V. Bernstein ([4]) and A. F. Leont' ev ([8]) it follows that a function $f(s)$ of class $B(r, h)$ is representable for $\sigma > 0$ by an absolutely convergent series

$$f(s) = \sum_{n=1}^{\infty} A_n(s), \tag{5}$$

where

$$A_k(s) = \sum_{\nu=0}^{p_k} \alpha_{k,\nu}[e^{-xs}, \lambda_{m_k}, \lambda_{m_k+1}, \dots, \lambda_{m_k+\nu}], \tag{6}$$

and the following properties hold: α) $p_k < r$; β) the sequence $\{\lambda_n\}$ satisfies condition (4); γ) there is a number $h_1 > 0$ such that $\lambda_{m_k+1} - \lambda_{m_k+p_k} \geq h_1$ for all k; δ) there is a number H such that for all k, $\lambda_{m_k+1} - \lambda_{m_k} < H$; ε) denote $\alpha_k = \max_\nu |\alpha_{k,\nu}|$, then $\alpha_k = O(e^{\varepsilon\lambda_{m_k}})$ for every $\varepsilon > 0$.

A function $f(s)$ of class $B(r,h)$ we shall also call a **series of class** $B(r,h)$, meaning the series (5).

Let $\lim_{k\to\infty} \alpha_k = \infty$. From properties a)–e) and geometric considerations it follows that there exists the least nonconvex (downward) envelope $C(x)$ of the points $P_k(\lambda_{m_k}, \log\alpha_k)$ (the possibility is not excluded that $\log\alpha_k = -\infty$), and $C(x)$ is a broken line with an infinite number of vertices at points of the form $P_{k_i}(\lambda_{m_{k_i}}, \log\alpha_{k_i})$. Denote $q_k = \exp C(\lambda_{m_k})$. Then for all k, $\alpha_k \leq q_k$, and for an infinite set of indices k_i,

$$q_{k_i} = \alpha_{k_i}. \tag{7}$$

Theorem 1. *Let D be a simply connected domain in which the function $f(s)$ of class $B(r,h)$ is regular.*

Then the family of functions $\{f_k(s)\}$, where

$$f_k(s) = \left[f(s) - \sum_{n=1}^{k-1} A_n(s)\right] q_k^{-1} \exp(\lambda_{m_k} s), \tag{8}$$

is uniformly bounded in every closed bounded domain belonging to D.

The proof scheme is the same as for Theorem 1 in ([1]), using a)–e) and the easily derived estimates:

$$|[e^{-xs}, \delta_0, \delta_1, \dots, \delta_n]| \leq |s|^n \exp(-\delta_0\sigma) \quad \text{for } \sigma > 0; \tag{9}$$

$$|[e^{-xs}, \delta_0, \delta_1, \dots, \delta_n]| \leq |s|^n \exp(-\delta_n\sigma) \quad \text{for } \sigma < 0. \tag{10}$$

Theorem 2. *Suppose that, under the conditions of Theorem 1, the imaginary axis is not a natural boundary of $f(s)$.*

Then every limit function $g(s)$ of the family $\{f_k(s)\}$ has the following properties:

a) $g(s)$ is analytic and single-valued in the domain consisting of the half-planes $\sigma > 0$, $\sigma < 0$, and the regular points of $f(s)$ on the imaginary axis.

b) For $\sigma > 0$, $g(s)$ is represented by a series

$$g(s) = \sum_{n=1}^{\infty} B_n(s)$$

of the same class $B(r, h)$ as $f(s)$, with bounded coefficients $\beta_{k,\nu}$.

c) A convergent expansion holds for $\sigma < 0$:

$$g(s) = \sum_{n=1}^{\infty} B_{-n}(s),$$

where

$$B_{-k}(s) = \sum_{\nu=1}^{u_{-k}} \beta_{-k,\nu}[e^{xs}, \mu_{-k,0}, \dots, \mu_{-k,\nu}], \quad \text{and}$$

$$|\beta_{-k,\nu}| \leq 1; \quad \mu_{-k,l+1} \leq \mu_{-k,l}; \quad \mu_{-(k+1),0} - \mu_{-k,u_{-k}} \geq h_1 > 0; \quad \mu_{-1,u_{-1}} \geq h_1 > 0.$$

d) Every isolated singularity of $f(s)$ on the imaginary axis is a simple pole of $g(s)$ or a regular point of $g(s)$.

The proof scheme is the same as for the corresponding theorems in ([1]), using (9) and (10).

Theorem 3. *If, under the conditions of Theorem 2, $g(s)$ is a limit function of the family $\{f_{k_i}(s)\}$, where $\{k_i\}$ satisfies (7), then $g(s)$ has singular points on the imaginary axis.*

Proof scheme. Suppose, on the contrary, that the theorem is false. Then $g(s)$ is an entire function. From b), c) of Theorem 2 and (9), (10) it follows that $|g(s)| < A(|s|+1)^r$ for $\sigma \geq 1$ and $\sigma \leq -1$, and $|g(s)| < A_1(|s|+1)^r \times |1 - \exp(h_1\sigma)|^{-1}$ for $|\sigma| \leq 1$. Considering the function $F(s) = g(s)\{1 - \exp[h_1(s - it_0 - 2\delta)]\}\{1 - \exp[h_1(s - it_0 + 2\delta)]\}$, with $\delta < \pi(8h_1)^{-1}$, first

in the rectangle ($|\sigma| \leq 1$, $|t - t_0| \leq 2\delta$), and then in the rectangle ($|\sigma| \leq 1$, $|t - t_0| \leq \delta$), it can be shown that $|g(s)| < A_2(|s|+1)^r$, and in the strip

$|\sigma| \leq 1$, and therefore $g(s)$ is a polynomial. After this, from b), taking (7) into account, one obtains:

$$g(s) = C_1 s^m + \cdots, \qquad C_1 \neq 0, \qquad m \geq 0,$$

and from c), as $\sigma \to -\infty$,

$$|g(\sigma)| < C_2 |\sigma|^r \exp(h_1 \sigma) \to 0.$$

The contradiction obtained proves Theorem 3.

Definition. We shall say that $f(s)$ has at the point s_0 a **singularity of type** k ($k \geq 0$), if

$$f(s) = a_k (s - s_0)^{-k} + \cdots + a_1 (s - s_0)^{-1} + p(s) \log(s - s_0) + f_1(s),$$

where $a_k \neq 0$, $p(s)$ is a polynomial, and $f_1(s)$ is regular at the point s_0.

Theorem 4. *If the sum of the series (5) of class $B(r, h)$ has on a segment of the imaginary axis of length greater than $2\pi r h^{-1}$ only isolated singularities of types 1 and 0, then the coefficients $a_{k,\nu}$ are bounded.*

The theorem is proved in the same way as the corresponding theorem in (2), using, instead of Pólya' s theorem, Leont' ev' s more general result ((8), pp. 48, 49), and also Theorem 3 of the present paper.

For a series $f(s)$ of class $B(r, h)$ with bounded coefficients, let us define the family of functions $\{f_x(s)\}$ ($x > 0$):

$$f_x(s) = \left[f(s) - \sum_{n=1}^{k(x)} A_n(s) \right] \exp(xs).$$

where $k(x)$ is such that

$$\lambda_{m_{k(x)}} < x \leq \lambda_{m_{k(x)}+1}.$$

For the family $\{f_x(s)\}$ and its limit functions $g(s)$, Theorem 1, Theorem 2 (with minor changes in c)), and also the following theorem hold.

Theorem 5. *If $g(s) = \lim f_{x_j}(s)$, then every isolated singularity $i\alpha$ of type 1 of $f(s)$ on the imaginary axis is a simple pole of $g(s)$, and*

$$\mathrm{Res}_g(i\alpha) = \mathrm{Res}_f(i\alpha) \cdot \lim_{x_j \to \infty} \exp(i\alpha x_j). \tag{11}$$

Sketch of the proof. First the theorem is proved for the case of a simple pole of $f(s)$—in the same way as in (3). In the general case, consider the function

$$\psi(s) = f(s) - \varphi(s),$$

where the series

$$\varphi(s) = \sum_{n=0}^{\infty} d_n \exp(-ns)$$

is chosen so that $\psi(s)$ has at the point $s = i\alpha$ a simple pole with the same residue as $f(s)$, and $d_n \to 0$ as $n \to \infty$. Then $g(s)$ coincides with a certain limit function $h(s)$ of the family $\{\psi_{x_j}(s)\}$, which has at the point $i\alpha$ a simple pole.

Theorem 6. *Let a function $f(s)$ of class $B(r, h)$, with exponents λ_n, have on a segment of the imaginary axis of length greater than $2\pi(D_\lambda + rh^{-1})$ only simple poles at the points $i\alpha_1, i\alpha_2, \dots, i\alpha_q$ (D_λ denotes the maximal density of $\{\lambda_n\}$, see (4)).*

Then:

a) *For some $\delta > 0$, $f(s)$ is regular and single-valued in the half-plane $\sigma > -\delta$ from which the set R of all singular points of $f(s)$ on the imaginary axis has been removed.*

b) *Every isolated point R is a simple pole of $f(s)$.*

c) *For every such isolated point $i\alpha$, the relation*

$$\alpha = m_1\alpha_1 + m_2\alpha_2 + \dots + m_q\alpha_q \tag{12}$$

holds, where $m_1, m_2, \dots, m_q$ are integers.

d) The numbers m_k in (12) can be chosen so that

$$m_1 + m_2 + \dots + m_q = 1. \tag{13}$$

Proof. a), b), c) are proved in general outline in the same way as in [3], using instead of Bernstein' s theorem the result of Leont′ev mentioned above.

Let us prove d). Let β be a real number which is not a linear combination of $\alpha_1, \dots, \alpha_q$ with rational coefficients. Applying c), already proved, to $f(s - i\beta)$, we obtain:

$$\alpha + \beta = n_1(\alpha_1 + \beta) + n_2(\alpha_2 + \beta) + \dots + n_q(\alpha_q + \beta). \tag{14}$$

Subtracting (12) from (14), we obtain $n_1 + n_2 + \dots + n_q = 1$ and

$$(n_1 - m_1)\alpha_1 + \dots + (n_q - m_q)\alpha_q = 0,$$

which together with (12) gives

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + \dots + n_q\alpha_q.$$

Theorem 7. Let a function $f(s)$ of the class $B(r, h)$ have, on the segment L_1 of length greater than $2\pi r h^{-1}$, only isolated singularities of type $\leq k$, and on the segment L of the imaginary axis of length greater than $4\pi r h^{-1}$ only isolated singularities $i\alpha_1, i\alpha_2, \dots, i\alpha_q$.

Then for every isolated singularity $i\alpha$ of $f(s)$ of type k on the imaginary axis, (12) and (13) hold.

Outline of the proof. Integrating $f(s)$ $k-1$ times, we obtain a series $\Phi(s)$ of the class $B(r, h)$ which, by Theorem 4, has bounded coefficients. Let $g(s)$ be an arbitrary limit function of the family $\{\Phi_x(s)\}$. Every singular point of type k of $f(s)$ on the imaginary axis is a point of type 1 of $\Phi(s)$ and, by Theorem 5, a simple pole of $g(s)$. In addition, on L, $g(s)$ has, by virtue of d) of Theorem 2, only simple poles. Application of Theorem 6 to $g(s)$ proves Theorem 7.

Remarks. Theorems 6 and 7 (with a certain refinement) make it possible to generalize Theorem 3 from [7] and the theorem (4) from [5] to broader classes of Dirichlet series. All the theorems from [6] can likewise be generalized.

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Note: Figure translations are in progress. See original paper for figures.