THEORY OF ELASTICITY

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Abstract

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ON THE EQUATIONS OF DYNAMICS OF A COMPRESSIBLE PLASTIC MEDIUM

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In investigating the problem of wave propagation in a plastic medium, the method by which the compressibility of this medium is taken into account is of substantial importance. In the case when the medium is under conditions of plane strain, the exact dynamic equations turn out to be complicated, and their analysis and solution are extremely difficult. This makes it expedient to introduce approximate, but simpler, models. The present note is devoted to the consideration of one approximate variant of the equations describing the plane motion of a plastic medium. Other variants were considered in ($^{1-3}$).

1. Let $\sigma_x, \sigma_y, \tau_{xy}$ be the components of the stress tensor; $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ the components of the strain tensor; u, v the components of the velocity vector. The equations determining the state of an incompressible Prandtl–Reuss elastic-plastic medium under conditions of plane strain consist of: the plasticity condition

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2; \tag{1}$$

the coaxiality condition

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\dot\gamma_{xy} - \dot\tau_{xy}/G}{\dot\varepsilon_x - \dot\varepsilon_y - (\dot\sigma_x - \dot\sigma_y)/2G} \tag{2}$$

and the incompressibility condition

$$\dot\varepsilon_x + \dot\varepsilon_y = 0$$

(cf. (^{4})). Let us make an additional assumption that allows the compressibility of the medium to be taken into account approximately (^{2}). Namely, suppose that instead of the equality $\dot\varepsilon_x + \dot\varepsilon_y = 0$ one may use a linear relation between the first invariants of the tensors

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot\varepsilon_x & \frac{1}{2}\dot\gamma_{xy} \\ \frac{1}{2}\dot\gamma_{xy} & \dot\varepsilon_y \end{pmatrix}$$

$$\dot{\varepsilon}_x + \dot{\varepsilon}_y = \frac{1}{2K}(\dot{\sigma}_x + \dot{\sigma}_y). \tag{3}$$

This assumption, of course, is meaningful only insofar as we remain within the framework of the plane problem.

The complete system of equations is obtained by adding the equations of motion, in which we shall not take account of convective terms, restricting ourselves to the case when changes in the dimensions of the body may be neglected.

Introducing the functions χ and φ, which identically satisfy the plasticity condition,

$$\left.\begin{matrix}\sigma_x\\ \sigma_y\end{matrix}\right\} = k(2\chi \mp \cos 2\varphi), \qquad \tau_{xy} = k \sin 2\varphi,$$

we arrive at the system of equations

$$\begin{gathered}
\frac{\partial \chi}{\partial x} - \sin 2\varphi \frac{\partial \varphi}{\partial x} + \cos 2\varphi \frac{\partial \varphi}{\partial y} - \frac{\rho}{2k}\frac{\partial u}{\partial t} = 0,\\
\frac{\partial \chi}{\partial y} + \cos 2\varphi \frac{\partial \varphi}{\partial x} + \sin 2\varphi \frac{\partial \varphi}{\partial y} - \frac{\rho}{2k}\frac{\partial v}{\partial t} = 0,\\
h\frac{\partial \chi}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad (4)\\
h_1 \frac{\partial \varphi}{\partial t} + \sin 2\varphi \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \cos 2\varphi \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0;
\end{gathered}$$

here ρ is the density, $h = 2k/K$, $h_1 = 2k/G$.

Let us consider the behavior of weak-discontinuity surfaces. For this purpose we apply to system (4) the theory of kinematic and dynamic compatibility conditions (^{5}).

Let the first derivatives of the unknown functions have a discontinuity on the surface $\omega(x, y, t) = 0$. The components of the discontinuity vector v_k satisfy the system of algebraic equations

$$\sum_{k=1}^{4} v_k \omega_{ik} = 0, \qquad i = 1, 2, 3, 4.$$

Equating to zero the determinant of this homogeneous system and introducing the angle α between the direction of the normal to the projection of the surface

Fig. 1

Figure 1: Fig. 1

$\omega = 0$ onto the plane $t = \text{const}$ and the x-axis, we obtain the equation for the propagation speed of the weak-discontinuity wave D:

Fig. 1

$$D = \pm\sqrt{\frac{c^2 + c_1^2}{2} \pm \sqrt{\frac{(c^2 + c_1^2)^2}{4} - c^2 c_1^2 \cos^2 2(\varphi - \alpha)}}\,, \tag{5}$$

where $c = \sqrt{2k/\rho h}, \quad c_1 = \sqrt{2k/\rho h_1}$.

As follows from formula (5), the propagation speed of the wave depends on the angle between the directions of propagation and of the principal normal stress. This circumstance was noted by G. A. Geniev (^{2}) for the case $h_1 = 0$. The dependence (5) is shown in Fig. 1a. As can be seen from the figure, the curve splits into two, which corresponds to the two types of waves that may occur in the model under consideration. When $\varphi - \alpha = 0$, these waves propagate with speeds $c = \sqrt{k/\rho}$ and $c_1 = \sqrt{G/\rho}$, so that it is natural to call them, respectively, a compression wave and a shear wave. It is interesting to note that when $h_1 > h$ the compression waves depend only weakly on the direction of propagation—the corresponding branch of the curve differs almost not at all from a circle—whereas the speed of the shear wave depends strongly on $\varphi - \alpha$ and, in particular, is equal to zero in the direction of the principal tangential stress.

To construct the characteristic surfaces it is important to know the equation of the Monge cone of system (4). It can be obtained by observing that the line of intersection of the cone and the plane $t = 1$ is the envelope of the family of straight lines (^{5})

$$x \cos\gamma + y \sin\gamma = D(\gamma), \qquad \gamma = \varphi - \alpha.$$

The form of this curve is shown in Fig. 1b.

It is essential to note that, since the velocity of wave propagation depends on the state of stress, shock waves may arise in a plastic medium. For a qualitative consideration, let us imagine that a weak-discontinuity wave propagates in some direction l with constant velocity, so that its graph in the coordinates l, t is a straight line. Since, upon passage of the wave, the state of stress changes, the diagram of Fig. 1a will rotate through some angle; hence the velocity of the following wave in the direction l will become different, and in the coordinates l, t we shall obtain a straight line with another slope. The family of such straight lines may have an envelope, which indicates the possibility of the occurrence of a shock wave. The construction of the front of this wave is possible only by solving system (4) with the use of the dynamic compatibility conditions on the wave front. This is a difficult problem, and therefore in a number of cases some benefit may be obtained from constructing the leading front of the wave under the assumption that it is the front of a weak discontinuity, which is considerably simpler.

Fig. 2

Let us consider, for example, the following problem. Suppose that on the segment AB of the surface of a semi-infinite body, compressed along the y-axis so that it is in a limiting state, a constant pressure p is instantaneously applied (Fig. 2). Let us find the leading wave front, assuming that $h_1 > h$ and that p is small. In this case the leading front will be a front of weak discontinuity. Since the compression wave propagates with the greater velocity, it will determine the shape of the leading front. But since the velocity of the compression wave depends only weakly on the direction of propagation, it may be regarded as approximately constant. As a result, the leading front is approximately determined as the envelope of the straight circular cones of equal opening with vertices at the points of the segment AB. The shear-wave front following it already has a considerably more complicated form.

2. Let us consider the particular case of system (4) for $h_1 = 0$, which corresponds to an elastic-plastic incompressible medium. From (5) it follows that only a shear wave arises, propagating with velocity $D = c_1 \cos 2(\varphi - \alpha)$. Along the directions of the principal shear stresses the velocity of propagation of the wave is equal to zero.

Of interest are self-similar motions of the medium. Introducing new variables $\lambda = x/t,\ \mu = y/t$, we obtain the system of equations

$$\frac{\partial \chi}{\partial \lambda} - \sin 2\varphi \frac{\partial \varphi}{\partial \lambda} + \cos 2\varphi \frac{\partial \varphi}{\partial \mu} + \frac{\rho}{2k}\left(\lambda \frac{\partial u}{\partial \lambda} + \mu \frac{\partial u}{\partial \mu}\right) = 0,$$

$$\frac{\partial \chi}{\partial \mu} + \cos 2\varphi \frac{\partial \varphi}{\partial \lambda} + \sin 2\varphi \frac{\partial \varphi}{\partial \mu} + \frac{\rho}{2k}\left(\lambda \frac{\partial v}{\partial \lambda} + \mu \frac{\partial v}{\partial \mu}\right) = 0, \tag{6}$$

$$\frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \mu} = 0,$$

$$h_1\left(\lambda \frac{\partial \varphi}{\partial \lambda} + \mu \frac{\partial \varphi}{\partial \mu}\right) - \sin 2\varphi \left(\frac{\partial u}{\partial \lambda} - \frac{\partial v}{\partial \mu}\right) + \cos 2\varphi \left(\frac{\partial u}{\partial \mu} + \frac{\partial v}{\partial \lambda}\right) = 0.$$

Investigation of system (6) shows that the angle between the direction of a characteristic and σ_1 satisfies the equation

$$\cos 2(\varphi + \alpha) = \pm r \frac{\sqrt{2}}{c_1} \sin(\beta - \alpha - \varphi); \tag{7}$$

Fig. 3

Figure 2: Fig. 3

(r, β) are polar coordinates in the λ, μ plane.

It follows from (7) that in the region bounded by the astroid

$$\begin{aligned} \frac{\lambda}{c_1} &= 2\cos(2\varphi+\gamma) - \cos\gamma\cos 2(\varphi+\gamma), \\ \frac{\mu}{c_1} &= -2\sin(2\varphi+\gamma) - \sin\gamma\cos 2(\varphi+\gamma), \end{aligned} \tag{8}$$

system (6) has 4 real and distinct characteristics, while outside this region it has 2. It is natural to regard the region inside the astroid as the region of subsonic motions. It is interesting to note that in gas dynamics the corresponding system of equations has the opposite properties, being hyperbolic in the supersonic region and elliptic in the subsonic region.

This circumstance can be interpreted geometrically in the following way. The astroid (8) is the curve obtained by intersecting Monge' s cone with the plane $t = 1$. In this plane, 4 distinct tangents can be drawn to the astroid from a point lying inside the curve, and 2 from a point lying outside it. In gas dynamics, correspondingly, either there is not a single real tangent if the point lies inside the circle, or there are 2 if the point lies outside it.

Hyperbolicity of system (6) in the subsonic region makes it possible to obtain the solution of a number of problems. Consider, for example, the following problem. Suppose that a thin blade is pressed into a plastic medium with a constant subsonic velocity. The problem in this case is self-similar. The values of the velocities and stresses compatible with the boundary conditions will be[4]

$$u = v = 0, \qquad \chi = -\frac{1}{2}, \qquad \varphi = \frac{\pi}{2}.$$

Fig. 3

The network of characteristics is shown in Fig. 3. This network obviously has the same form also in the case when a wedge of finite but small thickness is being pressed in.

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Note: Figure translations are in progress. See original paper for figures.