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Abstract

Full Text

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ON SOME INTEGRAL TRANSFORMATIONS IN EUCLIDEAN SPACE

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1. In the present paper four types of integral transformations with generalized kernel are considered in the n-dimensional Euclidean space R_n, and also in the space $\hat{R}_n$ of hyperplanes in R_n. Transformations of the first type transform a function of a point in R_n again into a function of a point and form a continuous group of transformations depending on one complex parameter λ; moreover this group includes the Laplace operator and the inverse operator to it. Transformations of the second type transform functions of hyperplanes into one another and also form a group analogous in its properties to the group of transformations of the first type. Transformations of the third and fourth types, also depending on the parameter λ, are pairwise mutually inverse and transform functions of a point into functions of a hyperplane and conversely. In particular, for a certain value of the parameter λ, consideration of a pair of such mutually inverse transformations makes it possible to write explicitly the solution of Radon' s problem of reconstructing a function from its known integrals over all possible hyperplanes (see ([1]), p. 107).

2. In order to obtain transformations of the first type, let us consider the space Ψ of basic functions, consisting of all infinitely differentiable functions $\psi(x)$ which at the origin are equal to zero together with all their derivatives and which decrease at infinity, together with all their derivatives, faster than any power of r^{-1}. In the space Ψ one can introduce a topology by specifying the countable family of norms

$$\|\psi\|_p = \sup_{q \le p} {}_x (r^{-p} + r^p)|D^q \psi(x)|,$$

$$p = 0, 1, 2, \dots; \quad r = \sqrt{x_1^2 + \dots + x_n^2}, \tag{1}$$

where D^q denotes the differentiation operator of order q with respect to arbitrary arguments. The topology introduced in this way turns Ψ into a complete countably normed space of type $K\{M_p\}$, where $M_p = r^{-p} + r^p$, and in the case $p = 0$ it is assumed that $M_0(0) = \infty$. It is easily verified that Ψ satisfies condition (P) (([2], p. 113) and therefore is a perfect space. It is easy to see that if $\psi(x) \in \Psi$, then also $r^\lambda \psi \in \Psi$. Hence it follows that the functional of the type of the function r^λ is a multiplier in the space Ψ' of generalized functions over Ψ. Let us introduce into consideration the space $\Phi = F^{-1}[\Psi]$, consisting of functions $\varphi(x)$ for which $\psi(x) \in \Psi$ are Fourier transforms. The space Φ consists of all infinitely differentiable functions decreasing at infinity, together with all their derivatives, faster than any power of r^{-1}, and orthogonal to all polynomials. It can be proved that Φ is a perfect space with differentiable shift. Thus, on the basis of the known theorem on convolutions (([2], p. 179), we obtain that the functional $R_\lambda = F^{-1}[r^\lambda]$ is a convolutor in the space Φ' of generalized functions over Φ. In other words, in the space Φ' an operation of convolution with the generalized function is defined.

$R_\lambda : f \to R_\lambda * f$. Let us find, in particular, $R_\lambda * R_\mu$. On the basis of the same convolution theorem we may write $F[R_\lambda * R_\mu] = F[R_\lambda] \cdot F[R_\mu] = r^\lambda \cdot r^\mu = r^{\lambda+\mu}$. Thus,

$$R_\lambda * R_\mu = R_{\lambda+\mu}, \qquad R_0 = F^{-1}[1] = \delta(x), \qquad R_\lambda * R_{-\lambda} = R_0 = \delta(x). \tag{2}$$

Relations (2) show that the operators $R_\lambda *$ form, with respect to the complex parameter λ, an additive group of transformations of functions of the space Φ'.

It is known that the operation of multiplication by r^{2k}, $k = 1, 2, ...$, in the space Ψ' is transformed under the Fourier transform into the operator $(-\Delta)^k$, where Δ is the Laplace operator. Thus, $R_{2k} = (-\Delta)^k \delta(x)$. From (4) we now obtain that $R_{-2k} *$ is the operator inverse to $(-\Delta)^k$. We see that the Laplace operator, its powers, and the operators inverse to them enter into the group of operators $R_\lambda *$. This circumstance gives grounds for calling this group the Laplace group.

Let us give an explicit expression for the generalized function R_λ. It may be obtained by a direct computation of the Fourier transform of the functional $r^\lambda \in \Psi'$.

$$R_\lambda(x) = \tag{3}$$

$$= \begin{cases} \dfrac{2^\lambda \Gamma((\lambda+n)/2)}{\pi^{n/2}\Gamma(-\lambda/2)}\, \eta^{-\lambda-n}, & \text{for } \lambda \neq 0, 2, \ldots \text{ and } \lambda \neq -n, -n-2, \ldots; \\ (-\Delta)^k \delta(x), & \text{for } \lambda = 2k;\ k = 0, 1, \ldots, \\ \dfrac{(-1)^k}{2^{n+2k-1}\pi^{n/2}\Gamma(n/2+k)\, k!}\, r^{2k} \ln r, & \text{for } \lambda = -n-2k;\ k = 0, 1, \ldots, \end{cases}$$

where the generalized functions $r^{-\lambda-n}$ and $r^{2k}\ln r$ act on Φ by the same formulas as on the space K of finite infinitely differentiable functions ([1]).

Let us note that in the case when $\operatorname{Re}\lambda < 0$ and f is a functional of the type of a locally integrable function, bounded in a neighborhood of the origin and decreasing at infinity faster than $r^{\operatorname{Re}\lambda}$, the transformation $f \to R_\lambda * f$ admits the integral representation:

$$R_\lambda * f = \int_{R_n} f(\xi) R_\lambda(x-\xi)\, d\xi. \tag{4}$$

3. We now pass to the consideration of transformations of the second type. Every hyperplane in R_n can be specified by a unit normal vector ω and by the distance ξ from the origin. Here ξ is regarded as positive if the direction from the origin to the plane coincides with the direction of the vector ω, and negative in the opposite case. Obviously, the parameters ω_i, ξ and $-\omega_i, -\xi$ specify one and the same plane. In view of this, as functions defined on $\hat{R}_n$, we shall consider even functions of the arguments ω_i, ξ*. Thus, let us consider the space $\hat{\Phi}$ of basic functions consisting of all even functions $\sigma(\omega, \xi)$, infinitely differentiable in all arguments, decreasing as $\xi \to \pm\infty$, together with all their derivatives, faster than any power of ξ^{-1}, and orthogonal to all polynomials in ξ in the sense of one-dimensional integration with respect to this argument. In the space $\hat{\Phi}'$ of generalized functions over $\hat{\Phi}$, one can define the operation of one-dimensional convolution with respect to the variable

* Taking into account here that, by virtue of the relation $|\omega| = \sqrt{\omega_1^2 + \dots + \omega_n^2} = 1$, not all of the n components ω_i are independent.

in the variable ξ with the functional $S_\lambda : g(\omega, \xi) \to S_\lambda * g$, where

$$S_\lambda(\xi) = \begin{cases} \dfrac{2^\lambda \Gamma((\lambda+1)/2)}{\sqrt{\pi}\Gamma(-\lambda/2)} |\xi|^{-\lambda-1} & \text{for } \lambda \neq 0, 2, \dots \text{ and } \lambda \neq -1, -3, \dots, \\ (-1)^k \delta^{(2k)}(\xi) & \text{for } \lambda = 2k;\ k = 0, 1, \dots, \\ \dfrac{(-1)^{k+1}}{\pi(2k)!} \xi^{2k} \ln|\xi| & \text{for } \lambda = -2k-1;\ k = 0, 1, \dots. \end{cases} \tag{5}$$

The operators $S_\lambda *$ satisfy relations analogous to (2). Therefore they form a continuous group of transformations of generalized functions of the space $\hat{\Phi}'$. Moreover, as is seen from (5), this group contains the operator $\partial^2/\partial\xi^2$, its powers, and the inverse operators to them. Here it is appropriate to note that the space $\hat{R}_n$ has a pronounced metric, as a result of which there is no Laplace operator in it in the usual sense of the word; however, many properties of the Laplace operator are possessed precisely by the operator $\partial^2/\partial\xi^2$. In particular, this operator commutes with the operators of transformation of functions $g(\omega, \xi)$

by means of motions of the space $\hat{R}_n$. We shall call the group of operators $S_\lambda *$ the Laplace group in the space $\hat{R}_n$. Below we shall see that the Laplace groups in the spaces R_n and $\hat{R}_n$ are closely connected with one another.

4. Let now $\varphi(x) \in \Phi$. Denote by $\rho(\omega, \xi)$ the integral of the function $\varphi(x)$ over the hyperplane $(\omega x) = \xi$, where $(\omega x) = \omega_1 x_1 + \cdots + \omega_{nx} n$. Then, as is not difficult to verify, $\rho(\omega, \xi) \in \hat{\Phi}$, and we can define the transformation

$$P_\lambda \times \varphi = S_\lambda * \rho(\omega, \xi), \tag{6}$$

where $S_\lambda * \rho$ denotes the one-dimensional convolution of the functional S_λ, considered in the preceding paragraph, with the basic function $\rho(\omega, \xi) \in \hat{\Phi}$ in the variable ξ. It can be shown that the operators $P_\lambda \times$ map the space Φ onto the whole space $\hat{\Phi}$.

Finally, let $\tau(\omega, \xi) \in \hat{\Phi}$. Consider the function

$$\varphi(x) = \frac{1}{(2\pi)^{n-1}} \frac{1}{2} \int_{\Omega_n} \tau(\omega, (\omega x))\, d\omega, \tag{7}$$

where Ω_n is the unit hypersphere in R_n, and $d\omega$ is the invariant measure on it. It is easy to see that $\varphi(x)$ is the integral, divided by $(2\pi)^{n-1}$, of the function $\tau(\omega, \xi)$ over the pencil of hyperplanes passing through the point $x \in R_n$. Putting now $\tau(\omega, \xi) = S_\lambda * \sigma(\omega, \xi)$, where $\sigma \in \hat{\Phi}$, we can define the operator $P_\lambda \circ$:

$$\sigma(\omega, \xi) \to \varphi(x) = P_\lambda \circ \sigma(\omega, \xi) = \frac{1}{(2\pi)^{n-1}} \frac{1}{2} \int_{\Omega_n} \tau(\omega, (\omega x))\, d\omega, \tag{8}$$

which, as is not difficult to verify, maps the space $\hat{\Phi}$ onto the whole space Φ.

We now define transformations $f(x) \to P_\lambda \times f$ and $g(\omega, \xi) \to P_\lambda \circ g$ of generalized functions of the spaces Φ' and $\hat{\Phi}'$ by means of the equalities

$$(P_\lambda \times f, \sigma) = (f, P_\lambda \circ \sigma); \qquad (P_\lambda \circ g, \varphi) = (g, P_\lambda \times \varphi). \tag{9}$$

In the case when the products $S_\lambda((\omega x)) f(x)$ and $S_\lambda(\xi) g(\omega, \xi)$ are functionals of the type of functions summable in the spaces R_n and $\hat{R}_n$, respectively, these transformations can be written in integral form

$$P_\lambda \times f = \int_{R_n} S_\lambda(\xi - (\omega x)) f(x)\, dx, \tag{10}$$

$$P_\lambda \circ g = \frac{1}{(2\pi)^{n-1}} \frac{1}{2} \int_{\Omega_n} \int_0^\infty S_\lambda((\omega x) - \xi) g(\omega, \xi)\, d\omega\, d\xi. \tag{11}$$

We note that in the case when f is a functional of the type of a function $f(x)$, integrable over each hyperplane $(\omega x) = \xi$, $P_0 \times f$ is the system of such integrals over all possible hyperplanes. We shall agree also in the general case, when f is an arbitrary generalized function, to call the functional $g(\omega, \xi) = P_0 \times f$ the **system of integrals of the generalized function f over all hyperplanes of the space R_n**. In an analogous manner, we shall call the generalized function $f(x) = P_0^\circ g$ the **averaging of the generalized function $g(\omega, \xi)$ over the pencils of hyperplanes passing through the (variable) point $x \in R_n$**.

5. We now establish the connection between transformations of all four types. First of all, from the very definition of the operators $P_\lambda \times$ and P_λ° it follows that

$$S_\lambda * (P_\mu \times f) = P_{\lambda+\mu} \times f, \qquad P_\lambda^\circ(S_\mu * g) = P_{\lambda+\mu}^\circ g. \tag{12}$$

Using the theorem on convolutions already mentioned, we find

$$P_\lambda^\circ(P_\mu \times f) = R_{\lambda+\mu-n+1} * f, \tag{13}$$

and, consequently, the operators P_λ° and $P_{-\lambda+n-1}\times$ are mutually inverse. This circumstance makes it possible to obtain from (13) and (12) the relations

$$R_\mu * (P_\lambda^\circ g) = P_{\lambda+\mu}^\circ g, \qquad P_\lambda \times (R_\mu * f) = P_{\lambda+\mu} \times f, \tag{14}$$

$$P_\lambda \times (P_\mu^\circ g) = S_{\lambda+\mu-n+1} * g, \tag{15}$$

as well as the relations

$$R_\mu * (P_\lambda^\circ g) = P_\lambda^\circ(S_\mu * g), \qquad P_\lambda \times (R_\mu * f) = S_\mu * (P_\lambda \times f), \tag{16}$$

which establish a connection between the Laplace groups in the spaces R_n and $\widehat{R}_n$. Let us note one particular case of formula (16):

$$\frac{\partial^2}{\partial \xi^2}(P_0 \times f) = P_0 \times (\Delta f). \tag{17}$$

6. Let us apply the theory set forth to the solution of Radon' s problem, which we formulate here as follows. Given a generalized function $g \in \widehat{\Phi}'$. It is required to find such a generalized function $f \in \Phi'$ that g be the system of its integrals over all hyperplanes of R_n. In other words, it is required to solve the equation

$$P_0 \times f = g. \tag{18}$$

The solution may be found with the aid of the operator P°_{n-1}, inverse to the operator $P_0\times$:

$$f = P^{\circ}_{n-1} g. \tag{19}$$

The obtained solution can be given another form. Introduce the notation $If = P^{\circ}_0(P_0 \times f)$. Obviously, If is the averaging, over all possible pencils, of the system of integrals of the generalized function f over the hyperplanes of R_n. Using (13) and (19), we obtain

$$f = R_{n-1} * (If). \tag{20}$$

If now n is odd, then $R_{n-1}* = (-\Delta)^{(n-1)/2}$, and we have:

$$f = (-\Delta)^{(n-1)/2}(If). \tag{21}$$

Let us note here that the formula $f = P^{\circ}_{n-1-2k}(P_{2k} \times f)$ gives a solution of the problem of reconstructing the function f from the known normal derivatives of order $2k$ of its integrals over all possible hyperplanes.

7. In conclusion, let us observe that an analogous theory can also be constructed in the pseudo-Euclidean space $R_n^{(q)}$ of index q.

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2. I. M. Gel' fand, G. E. Shilov, *Spaces of Test and Generalized Functions*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.