

ON THE STABILITY OF ONE CASE OF A SPHERICAL CONVERGING SHOCK WAVE

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Abstract

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HYDROMECHANICS

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ON THE STABILITY OF ONE CASE OF A SPHERICAL CONVERGING SHOCK WAVE

(Presented by Academician A. D. Sakharov on 16 VI 1960)

1. We consider the stability of a strong spherical converging shock wave in an ideal gas with isentropic exponent $\gamma = 7$. The unperturbed motion is taken in the form (1)

$$V_r = \frac{1}{10} \frac{r}{t}; \quad c = -\frac{\sqrt{21}}{10} \frac{r}{t}; \quad p = \frac{1}{25} \rho_0 \frac{r^2}{t^2} \frac{\xi}{\xi_0}; \quad \rho = \frac{4}{3} \rho_0 \frac{\xi}{\xi_0}, \quad (1)$$

where $\xi = r/t^{2/5}$; the trajectory of the shock front is $R(t) = \xi_0 t^{2/5}$; ξ_0 is a certain constant; $R \leq r \leq \infty$; $-\infty \leq t \leq 0$.

A particular solution of the linearized equations of hydrodynamics has the form

$$\begin{aligned} V_r' &= t^\lambda \frac{r}{t} \sum_{\alpha=1}^4 a_\alpha D_\alpha e^{\mu_\alpha z} Y_n(\theta, \varphi), \\ V_\theta' &= t^\lambda \frac{r^2}{t} \sum_{\alpha=1}^4 b_\alpha D_\alpha e^{\mu_\alpha z} \frac{1}{r} \frac{\partial Y_n(\theta, \varphi)}{\partial \theta}, \\ V_\varphi' &= t^\lambda \frac{r^2}{t} \sum_{\alpha=1}^4 b_\alpha D_\alpha e^{\mu_\alpha z} \frac{1}{r \sin \theta} \frac{\partial Y_n(\theta, \varphi)}{\partial \varphi}, \\ p' &= \frac{4}{3} \rho_0 t^\lambda \frac{r^2}{t^2} \sum_{\alpha=1}^4 c_\alpha D_\alpha e^{(\mu_\alpha+1)z} Y_n(\theta, \varphi), \\ \rho' &= \frac{4}{3} \rho_0 t^\lambda \sum_{\alpha=1}^4 D_\alpha e^{(\mu_\alpha+1)z} Y_n(\theta, \varphi), \end{aligned} \quad (2)$$

where $Y_n(\theta, \varphi)$ is a spherical function of order n ; $z = \ln(\xi/\xi_0)$; the exponents μ and λ are related by the equation

$$\begin{vmatrix} \lambda - \frac{3}{10}\mu - \frac{4}{5} & 0 & 3 + \mu & -\frac{9}{100} \\ 0 & \lambda - \frac{3}{10}\mu - \frac{4}{5} & 1 & 0 \\ 4 + \mu & -n(n+1) & 0 & \lambda - \frac{3}{10}\mu \\ -4 & 0 & \frac{100}{3} \left(\lambda - \frac{3}{10}\mu \right) & -7 \left(\lambda - \frac{3}{10}\mu \right) \end{vmatrix} = 0. \quad (3)$$

The coefficients $a_\alpha, b_\alpha, c_\alpha$ ($\alpha = 1, 2, 3, 4$) are the algebraic complements of the first three elements of the last row of determinant (3), divided by the algebraic complement of the fourth element; finally, the constants

D_α satisfy the system

$$\begin{aligned} \sum_{\alpha=1}^4 (1 - 10b_\alpha) D_\alpha &= 0, & \sum_{\alpha=1}^4 \left(a_\alpha - \frac{5}{2} \lambda b_\alpha \right) D_\alpha, \\ \sum_{\alpha=1}^4 \left[\frac{2}{3} c_\alpha + \left(\lambda - \frac{1}{5} \right) b_\alpha \right] D_\alpha &= 0. \end{aligned} \quad (4)$$

2. Solving the Routh-Hurwitz problem (2) for equation (3), in the plane of the parameter λ we find the regions qualitatively depicted in Fig. 1, where the first and second numbers denote the number of roots of equation (3) with $\text{Re } \mu < 0$ and $\text{Re } \mu > 0$, respectively. The distribution of the roots can be obtained by knowing the direction in which the curves are traversed (when the imaginary axis of the μ -plane is traversed from bottom to top) and the number of roots with $\text{Re } \mu < 0$ in one of the regions, for example at

Fig. 1

$$\text{Re } \lambda = \pm\infty.$$

All the curves have vertical asymptotes, the abscissae of which do not depend on the number n :

$$\text{Re } \lambda = \frac{2}{5} \pm \frac{3}{10} \sqrt{21}, \quad \text{Re } \lambda = 0, \quad \text{Re } \lambda = \frac{4}{5}. \quad (5)$$

If bounded initial perturbations are prescribed in the whole region $0 \leq z \leq \infty$ and do not vanish as $z \rightarrow \infty$, then on the left boundary of the regions 3, 1 and 4, 0 (respectively 2, 1 and 3, 0 for $n = 0$) the solution of the Cauchy problem by means of the inverse Laplace transform loses its meaning. Expanding the initial

perturbations in a series in powers of z , we construct an analytic continuation of the solution. The function that realizes this continuation has, in the λ -plane, a pole corresponding to the point of the boundary mentioned for which $\mu_4 = 0$. In the case when the initial perturbations are $\sim \exp(-\alpha z)$, where $\alpha > 0$, the analytic continuation of the solution will have a pole at the point of the λ -plane for which $\mu_4 = -\alpha$. In this way, for different initial data, the lower boundary of instability is determined.

If, however, the initial perturbations are localized in a finite region, then the contour of the inverse Laplace transform can be moved beyond the boundary of the region 3,1 arbitrarily far to the right. The behavior of the solutions as $t \rightarrow 0$ is determined by the zeros of the determinant of system (4), when $D_4 = 0$. Among the zeros of this determinant there will also be branch points of the function μ ; however, in the region 3,1 they should not be taken into account, since the Laplace transforms are symmetric with respect to permutation of the branches μ_i ($i = 1, 2, 3$). The branch points of the function $\mu(\lambda)$ are computed from the formulas:

$$\lambda = -\frac{1}{2} \pm \frac{4}{7} \sqrt{x + \frac{4}{25} \left[1 + \left(\frac{0.09}{x} \right)^2 N \right]},$$

where $N = n(n+1)$, and x satisfies the equation

$$\frac{400}{3}x^5 + (49N+5)x^4 + 13 \cdot \frac{9}{25}Nx^3 + \frac{24}{9} \left(\frac{9}{25} \right)^2 Nx^2 - \frac{1}{4} \left(\frac{9}{25} \right)^3 N^2x - \frac{1}{25} \left(\frac{9}{25} \right)^3 N^2 = 0,$$

since the dependence $\mu(\lambda)$ can be represented in parametric form

$$\left(\lambda - \frac{3}{10}\mu \right) \left(\lambda - \frac{3}{10}\mu - \frac{4}{5} \right) = x,$$

$$\frac{100}{3}x^2 - [7(\mu+3)^2 - (7N+3)]x + \frac{9}{25}N = 0. \quad (6)$$

The equation for determining the poles, after extracting the factor $(\mu_1 - \mu_2)(\mu_1 - \mu_3)(\mu_2 - \mu_3)$, becomes a symmetric function of μ_1, μ_2, μ_3 . By virtue of equation (3), it can be replaced by a rational function of μ_4 and, with the aid of formulas (6), represented in the form

$$\begin{vmatrix} \left[\frac{4}{25} - \left(\frac{9}{100} \right)^2 \frac{N}{x} \right] y & -\frac{4}{35}y - \frac{82}{175} - \left(\frac{9}{100} \right)^2 \frac{N}{x} & \frac{7}{2}P - y - \frac{13}{10} \\ -\frac{7 \cdot 83}{30} & 14P + \frac{292}{21} & \frac{223}{4} - \frac{5 \cdot 49}{2}P \\ -\left(3P + \frac{13}{10} \right) & 2 & 0 \end{vmatrix} = 0,$$

where

$$y = \sqrt{x + \frac{4}{25}}, \quad P = y - Q, \quad Q = \sqrt{\frac{1}{7} \left[3x + \frac{63N + 27}{100} + \frac{81N}{2500x} \right]}.$$

In the case $n = 0$, the growth of perturbations is determined by the pole

$$\lambda_0 = -\frac{3}{70}(21 - \sqrt{21}) \simeq -0.7036$$

and as $t \rightarrow 0$ has the form t^{λ_0} . For $n \gg 1$ we have the pole

$$\lambda_1 = -\frac{234}{245} \simeq -0.9551.$$

3. To the solution found in (2), for which $(\text{rot } \mathbf{V}')_r = 0$, one should add a vortex wave

$$V_\theta'' = -\frac{A(r, t)}{r} \frac{1}{\sin \theta} \frac{\partial Y_n}{\partial \varphi}, \quad V_\varphi'' = \frac{A(r, t)}{r} \frac{\partial Y_n}{\partial \theta}, \quad V_r'' = p'' = \rho'' = 0,$$

where the quantity $A(r, t)$ is conserved along particle trajectories, and $\text{div } \mathbf{V}'' = 0$. After this, the system of angular functions in which the vector \mathbf{V}' is expanded will be complete [3]. The added solution does not affect the shock-wave front, regardless of whether it arose before or after the passage of the wave front.

4. The stability problem has several trivial solutions, which are obtained by varying the parameters of the unperturbed solution (1): the focusing point $r_\phi = 0$, the focusing time $t_\phi = 0$, and the front velocity ξ_0 . These solutions are contained in formulas (2), (4): the first for $n = 1$ and $\lambda = -\frac{2}{5}$, the second for $n = 0$ and $\lambda = -1$, and lead to formal instability; finally, the third—for $n = 0$ and $\lambda = 0$.

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