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Abstract

Full Text

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CHAPLYGIN' S THEOREM FOR ELLIPTIC EQUATIONS

(Presented by Academician S. L. Sobolev on 23 IV 1960)

In the present article we shall consider some basic properties of Green's function, Chaplygin' s theorem, the limits of its applicability, and comparison theorems. In what follows, everywhere, unless the contrary is stated, the notation and definitions from ([1]) are used.

We shall need the following.

Remark. The results of the author' s work ([3]) are based on general properties of integral equations and therefore, without substantial changes, carry over to the Ries–Schauder equations in the space $\mathcal{L}_2$ ([2]). Let us also recall that an integral operator with a weak singularity is completely continuous in $\mathcal{L}_2$ ([2]).

Let in the domain T a linear differential operator of second order of elliptic type be given:

$$\mathfrak{M} = \sum_{i,k=1}^{m} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial x_i} + c.$$

We shall assume that the operator $\mathfrak{M}$ is self-adjoint. Further, with respect to the domain T and the operator $\mathfrak{M}$ we shall consider as satisfied one or another set of conditions ensuring the existence of the Green' s function $F = F(x, y)$ of the Dirichlet problem

$$\mathfrak{M}u = -f, \qquad f \geqslant 0, \tag{1}$$

$$u|_{\mathfrak{S}T} = \varphi, \qquad \varphi \geqslant 0. \tag{2}$$

If $c \leqslant 0$, then regular solutions of equation (1) possess the property of a maximum (minimum) (([1]), Theorems 3, I and 3, II).

Definition. The coefficient $c(x)$ will be called **critical** for a given domain T and, conversely, the domain T will be called **critical** with respect to $c(x)$, if for these c and T the uniqueness theorem 21, II ([1]) is violated. Critical values $c(x)$ will be denoted by $\lambda = \lambda(x)$, and domains by $T = T(c)$.

We shall call λ_1 the **least** or **first critical value**, if there exists no λ such that $\lambda \leqslant \lambda_1$, $\lambda \neq \lambda_1$ in the domain T. $T_1(c)$ is defined analogously. The totality of first critical values forms a set of functions Λ_1, whose elements have the property that all of them intersect one another.

We shall say that $c < \lambda_1$ if there are no critical values $\lambda \leqslant c$. Denote the set of all critical values of c for the given domain T by Λ. λ_k is defined as the least among the critical values

$$\Lambda - \sum_{i=1}^{k-1} \Lambda_i.$$

Here "least" is understood in the sense of the definition of λ_1.

The meaning of the notations and inequalities $T_k(c)$, $T(c)$, $\lambda_{k-1} < c < \lambda_k$, $T_{k-1} < T < T_k$ is clear from the preceding definitions.

Theorem 1. If $c < \lambda_1$, then the Green' s function of the Dirichlet problem is positive in the domain T: $F \geqslant 0$.

Proof. Consider two cases.

1°. $c \leqslant 0$. It is known [(1)] that for $c \leqslant 0$ the Green' s function F exists. Further,

$$\begin{aligned} \mathfrak{M}_x F = 0 \quad & \text{for} \quad x \in T - \mathfrak{F}T - y; \\ F = 0 \quad & \text{for} \quad x \in \mathfrak{F}T, \quad y \in T - \mathfrak{F}T; \\ F(x,y) \to \infty \quad & \text{as} \quad x \to y. \end{aligned} \tag{3}$$

Exclude some neighborhood $O(y)$ of the point y, where $F > 0$. In the domain $T - O(y)$, F is a regular solution of equation (3), which assumes nonnegative values on the boundary. From the minimum property (([1]), Theorems 3, I and 3, II) it follows that $F \geqslant 0$ in $T - O(y)$, and hence also in the entire domain T.

2°. c is arbitrary. Choose a function $\chi \in C^{(0,\alpha)}$ such that $c - \chi < 0$, $\chi \geqslant 0$. Construct the Green' s function F^* for the operator $\mathfrak{M} - \chi$, which, by 1°, will be positive. Then F is determined from the equation (([1]), pp. 82–83)

$$F(x,y) = F^*(x,y) + \int_T F^*(x,t)\chi(t)F(t,y)\,dt. \tag{4}$$

By the hypothesis of the theorem $c < \lambda_1$. It follows that $1 < \mu_1$, where μ_1 is the first eigenvalue of equation (4).

Indeed, by the definition of λ_1, the Green' s function F exists and is unique for every $c < \lambda_1$. On the other hand, it is a solution of equation (4). If we assume that $\mu_1 \leqslant 1$, then for the existence and uniqueness of a solution for $\mu = \mu_1$ it is necessary that the free term F^* be orthogonal to all eigenfunctions of the kernel

$K = F^*\chi$ corresponding to μ_1. But $F^* \geqslant 0$, $\chi \geqslant 0$, hence $K \geqslant 0$, and among the eigenfunctions there will be one positive. This leads to a contradiction, since F^* is also positive.

Fredholm theory is applicable to equation (4).

From Theorem 1 ([3]) and the remark it follows that $F \geqslant 0$ in the domain T. We note that if $c > \lambda_1$, then for any choice of χ we shall have $1 > \mu_1$, and from Theorem 1 ([3]) it follows that the Green' s function cannot be positive everywhere in the domain T.

Theorem 2. For $c < \lambda_1$, Chaplygin' s theorem holds: if $f \geqslant 0$, $\varphi \geqslant 0$, then the solution of equation (1)–(2) is positive, $u \geqslant 0$, in the domain T.

Indeed, the solution u can be represented as the sum

$$u = u_1 + u_2,$$

where u_1 is the solution of the problem

$$\mathfrak{M}u_1 = -f, \qquad u_1|_{\mathfrak{F}T} = 0,$$

and u_2 is the solution of the problem

$$\mathfrak{M}u_2 = 0, \qquad u_2|_{\mathfrak{F}T} = \varphi.$$

1) $u_1 \geqslant 0$, since, by Theorem 1, from the condition $c < \lambda_1$ it follows that $F \geqslant 0$, and

$$u_1 = \int_T F f \, dy.$$

2) $u_2 \geqslant 0$: suppose the contrary. Then in some domain $T_1 \subset T$ we have

$$\mathfrak{M}u_2 = 0, \qquad u_2|_{\mathfrak{F}T_1} = 0, \qquad u_2 \leqslant 0, \qquad u_2 \neq 0.$$

This cannot be, since the domain T, and hence also T_1, is smaller than the critical one.

Corollary. We obtain the obvious generalization of the theorem on maximum (minimum) values: for $c < \lambda_1$, the regular solution of the Dirichlet problem (1) –(2) has no points of negative relative minimum in the domain $T - \mathfrak{F}T$.

Let F and F_1 be the Green' s functions corresponding to c and c_1, respectively, where $c_1 \leq c$ and $F \geq 0$. Then it is not difficult to see that $F_1 \geq 0$ as well, and the difference $F - F_1$ satisfies the relation

$$\mathfrak{M}_x(F - F_1) = -(c - c_1)F_1,$$

$$F - F_1 = 0 \quad \text{for } x \in \mathfrak{F}T, \quad y \in T - \mathfrak{F}T.$$

Hence

$$F - F_1 = \int_T F(c - c_1)F_1 \, dt \geq 0.$$

The requirement $F \geq 0$ is equivalent to $c < \lambda_1$.

Thus we have proved the following.

Theorem 3. *The Green' s function is a monotonically increasing function of the coefficient c, provided that c, while varying, remains below the critical value λ_1.*

From this it follows:

Theorem 4 (comparison theorem). *If $c_1 \leq c < \lambda_1$, then $u_1 \leq u$, where u, u_1 are the solutions of problem (1)–(2) corresponding to c, c_1.*

Next we shall consider the dependence of the Green' s function on the domain. Let the operator $\mathfrak{M}$ be defined in the domain T, $T_1 \in T$, and let F_1 be the Green' s function of the Dirichlet problem for the domain T_1. Suppose, moreover, that the domain T is smaller than critical. Then the functions F and F_1 exist and, by Theorem 1, are positive. Further, under these conditions there exists the principal fundamental solution G of the equation $\mathfrak{M}u = 0$ (([1], pp. 69–76), and the Green' s function for any domain T has the form

$$F = G + g,$$

where g is the regular solution of the equation $\mathfrak{M}g = 0$, $g = -G$ for $x \in \mathfrak{F}T$ (([1], p. 81).

Therefore the difference $F - F_1 = g_1$ is a regular solution of the equation $\mathfrak{M}g_1 = 0$, $g_1|_{\mathfrak{F}T_1} \geq 0$. By Theorem 2 we have $g_1 \geq 0$, i.e. $F \geq F_1$ in the domain T_1.

We have proved:

Theorem 5. *The Green' s function is a monotonically increasing function of the domain T, provided that T, while varying, remains smaller than critical.*

Hence it follows:

Theorem 6 (comparison theorem with respect to the domain). *Let u, u_1 be solutions of the Dirichlet problem for the domains T, T_1; $u|_{\mathfrak{F}T} = 0$,*

$u_1|_{\mathfrak{F}T_1} = 0$, *and let the domain* T *be smaller than critical. Then* $u \geq u_1$ *in the domain* T_1.

Theorems 3 and 5 give broad scope for estimating the Green' s function from below and from above, while Theorems 2, 4, and 6 give such estimates for solutions.

We note that the results set forth, with certain changes, carry over also to non-self-adjoint equations.

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CITED LITERATURE

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[2] S. G. Mikhlin, *Lectures on Linear Integral Equations*, 1959.
[3] A. G. Teterev, Reports of the Scientific Conference on Theoretical and Applied Problems of Mathematics and Mechanics, Tomsk, 1960.

Note: Figure translations are in progress. See original paper for figures.