MATHEMATICS

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Abstract

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PERTURBATION OF EIGENVALUES AND EIGENELEMENTS FOR CERTAIN NON-SELF-ADJOINT OPERATORS

In ([1]) the authors investigated the perturbation of solutions of certain linear equations—algebraic and differential. In the present note, analogous methods are used to study the question of the perturbation of eigenvalues and elements (in the non-self-adjoint case)*.

1. Algebraic case. Let A_0, A_1, and $A_\varepsilon = A_0 + \varepsilon A_1$ be n-dimensional matrices (generally speaking, non-Hermitian), and let $\lambda = 0$ be an eigenvalue of A_0; to it corresponds an invariant space S_0 of dimension N—the linear span of the eigenvectors and associated vectors corresponding to the eigenvalue $\lambda = 0$.

As is known, for the matrix A_0 there exists a Jordan basis $\{x_{ij}\}$ corresponding to the eigenvalue $\lambda = 0$, where $\{x_{i0}\}$ $(i = 1, 2, \dots, r)$ is a complete system of eigenvectors corresponding to the eigenvalue $\lambda = 0$: $A_0 x_{i0} = 0$; x_{ij} $(j = 1, 2, \dots, n_i - 1)$ are the associated vectors corresponding to x_{i0} (of the j-th order), forming a Jordan chain of length n_i and satisfying the relations $A_0 x_{ij} = x_{i,j-1}$ (for $j = 1, 2, \dots, n_i - 1$).

Let E_{i0} be the linear span of all eigenvectors x_{i0} to which there correspond Jordan chains of one and the same length n_i, $n_1 > n_2 > \dots > n_l > 1$; p_i is the dimension of E_{i0}; E_{ij} $(j = 1, 2, \dots, n_i - 1)$ is the linear span of the eigenvectors x_{ij} of the j-th order associated with x_{i0}. Obviously, the dimension N of the space S_0 is equal to

$$N = \sum_{i=1}^{l} p_i n_i$$

and

$$S_0 = \sum_{i=1}^{l} \sum_{j=0}^{n_i - 1} E_{ij}.$$

Theorem. Under the described structure of the invariant space S_0 corresponding to the eigenvalue $\lambda = 0$ of the matrix A_0, the matrix A_ε has $N = \sum_{i=1}^{l} n_i p_i$ eigenvalues λ_ε tending to zero as $\varepsilon \to 0$; among them, $n_i p_i$ eigenvalues are expressed by power series of the form

$$\lambda_\varepsilon = \sum_{k=1}^{\infty} \lambda_k \varepsilon^{k/n_i}, \tag{1}$$

* The contents of both notes were reported at the Fifth All-Union Conference on Functional Analysis in Baku in October 1959.

where $\lambda_k = \lambda_k(q)$ $(q = 1, 2, \dots, n_i p_i)$, and they correspond to the eigenvectors

$$v_\varepsilon = \sum_{k=0}^{\infty} v_k \varepsilon^{k/n_i}. \tag{2}$$

Here $v_0 = \sum_{j=1}^{i} v_{j0}$, $v_{j0} \in E_{j0}$; v_{i0} and $\lambda_1^{n_i}$ are the eigenvector and eigenvalue, respectively, of a certain operator C_i, acting from E_{i0} into E_{i0}, while v_{j0} are determined from v_{i0} by applying a fixed linear operator B_{ij}, acting from E_{i0} into E_{j0}.

In the formulation of the theorem it must be stipulated that each operator C_i has nonzero and distinct eigenvalues. The matrix of the operator C_1 has the form $C_1 = \|(A_1 y_i, z_j)\|_{i,j=1,\dots,p_1}$, where y_i $(i = 1, \dots, p_1)$ form a basis in E_{10}, and z_j $(j = 1, \dots, p_1)$ form the corresponding basis in the subspace E_{10}^*, which plays the analogous role for the adjoint matrix A_0^*. Similarly, on the basis of a certain recurrent process, $C_2, \dots, C_l$, as well as B_{ij}, are constructed.

The coefficients λ_k, v_k $(k = 1, 2, \dots)$ of the expansions (1), (2) are obtained by successively solving the system of equations

$$A_0 v_0 = 0; \tag{3}$$

$$A_0 v_k = b_k = \begin{cases} \displaystyle\sum_{j=1}^{k} \lambda_j v_{k-j}, & \text{for } k < n_i, \\ \displaystyle\sum_{j=1}^{k} \lambda_j v_{k-j} - A_1 v_{k-n_i}, & \text{for } k \geq n_i, \end{cases} \tag{4}$$

obtained as a result of comparing coefficients in the equality $(A_\varepsilon - \lambda_\varepsilon I) v_\varepsilon = 0$ after substituting into it the expansions (1), (2). At the same time, since we are on the spectrum, the solvability conditions for equations (4) are used, as a result of which v_k and λ_{k+1} are determined at the $(n_i + k)$-th step of the iteration.

In the simplest case where there is one Jordan block, the expansions of λ_ε and v_ε will be the same as those given below in § 2. The theorem also remains valid if the matrix A_ε is expanded in ε into a convergent power series.

2. Let us now consider the case of differential operators, restricting ourselves, for simplicity, to operators of second order and to their linear dependence on ε. Let L_0 be an elliptic (generally speaking, non-self-adjoint) operator of second order; L_1, an operator of order not higher than the second; $L_\varepsilon = L_0 + \varepsilon L_1$, and suppose these operators act on functions u defined in a domain D with boundary Γ and vanishing on Γ: $u|_\Gamma = 0$. The coefficients of L_0 and L_1 are continuous in $D + \Gamma$, and the boundary Γ is smooth.

Suppose that, under these boundary conditions, the operator L_0 has $\lambda = 0$ as an eigenvalue. As is known, by a Jordan chain $u_0, u_1, \dots, u_{n-1}$, corresponding to the eigenvalue $\lambda = 0$, one understands successive solutions of the problems

$$\begin{gathered} L_0 u_0 = 0 \qquad u_0|_\Gamma = 0; \\ L_0 u_i = u_{i-1}, \qquad u_i|_\Gamma = 0 \quad (i = 1, \dots, n-1). \end{gathered} \tag{5}$$

The number of elements of such a chain $\{u_i\}$ is finite.

The theorem formulated above for matrices turns out to be valid in our case as well. We shall present the construction of the expansion of the eigenvalues λ_ε of the operator L_ε and of its eigenfunctions v_ε in the case when there is only one Jordan chain and its length is equal to n. As

and above, we shall seek the expansions of λ_ε and v_ε in the form

$$\lambda_\varepsilon = \sum_{k=1}^{\infty} \lambda_k \varepsilon^{k/n}, \qquad v_\varepsilon(x) = \sum_{k=0}^{\infty} v_k(x) \varepsilon^{k/n}. \tag{6}$$

Substituting these expansions into the equation $((L_0 + \varepsilon L_1) - \lambda_\varepsilon I) v_\varepsilon = 0$, we obtain the system of equations

$$L_0 v_0 = 0; \qquad v_0|_\Gamma = 0; \tag{7}$$

$$L_0 v_k = b_k, \qquad b_k = \begin{cases} \sum_{j=1}^{k} \lambda_j v_{k-j}, & \text{for } 0 < k < n, \\ \sum_{j=1}^{k} \lambda_j v_{k-j} - L_1 v_{k-n}, & \text{for } k \geq n, \end{cases} \tag{8}$$

$$v_k|_\Gamma = 0 \qquad (k = 1, 2, \dots).$$

Denote by u_0 and z_0 the eigenfunctions of the given and adjoint operators:

$$L_0 u_0 = 0, \qquad u_0|_\Gamma = 0; \qquad L_0^* z_0 = 0, \qquad z_0|_\Gamma = 0. \tag{9}$$

Then, as is known, the solvability condition for problem (8) is the fulfillment of the equality

$$\iint_D b_k z_0 \, dx = (b_k, z_0) = 0. \tag{10}$$

It is seen from (7) that $v_0 = u_0$ (or Cu_0, $C = \text{const}$); the condition (10) for $k = n$ (if u_0 is normalized accordingly and it is taken into account that $(v_{k-j}, z_0) = 0$ for $n - 2 \geq j > 0$) leads to the equation

$$\lambda_1^n - (L_1 u_0, z_0) = 0. \tag{11}$$

We make the assumption

$$(L_1 u_0, z_0) = C \neq 0.$$

Thus from (11) one determines n distinct values $\lambda_1 = \sqrt[n]{C}$. Choose one of them and denote it by λ_1. Then from (8), for $k = 1$, we find v_1. Further, using successively (10) and (8), we find $\lambda_2, v_2, \ldots, \lambda_p, v_p, \lambda_{p+1}, \ldots$

The proof of convergence, as in the algebraic case, reduces to the construction of a corresponding majorant. Under the corresponding conditions, the iterative process also works in the general case.

3. In ([2]), § 9, for the self-adjoint case, the perturbation of eigenvalues and eigenfunctions is studied when the perturbed operator is of higher order than the unperturbed one. In this case boundary-layer phenomena arise. In the non-self-adjoint case, when the perturbed operator is of higher order than the unperturbed one, the expansion for the eigenvalues is analogous to formula (6), while in the expansion for the eigenfunctions additional terms of boundary-layer type will appear, as in ([2]).

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[1] M. I. Vishik, L. A. Lyusternik, DAN, 129, No. 6 (1959). [2] M. I. Vishik, L. A. Lyusternik, Uspekhi matem. nauk, 12, issue 5 (77), 3 (1957).

Note: Figure translations are in progress. See original paper for figures.