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Abstract

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MATHEMATICS

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On the General Theory of Biorthogonal Kernels

In a note by one of the authors (1), questions concerning quasi-isometric mappings of Hilbert spaces of functions L_σ^2 were studied and, in particular, an analytic characterization of such mappings was given, generalizing the classical Bochner theorem (2). The present note is essentially a continuation of that investigation. Here the notions of Bessel and Hilbert kernels are introduced and some of their properties are established, which are natural continual analogues of results of N. K. Bari (3–5).

Let $\sigma_k(x)$ $(k = 1, 2)$ be a nondecreasing function, defined and continuous from the right on the interval (a_k, b_k), where $-\infty \leq a_k < b_k \leq +\infty$, and of bounded variation on every segment $[\alpha, \beta] \subset (a_k, b_k)$. Denote by $H_k = L^2_{\sigma_k}(a_k, b_k)$ the Hilbert space of all σ_k-measurable functions on (a_k, b_k) that are square-summable.

As shown in (1), if the operators R_1 and R_{*1}, mapping all of H_1 onto all of H_2, form an isometric pair, i.e.

$$(R_1 f_1, R_{*1} f_2)_{\sigma_2} = (R_{*1} f_1, R_1 f_2)_{\sigma_2} = (f_1, f_2)_{\sigma_1} \quad (f_1, f_2 \in H_1),$$

where

$$(f, g)_{\sigma_k} = \int_{a_k}^{b_k} f(x)\overline{g(x)}\, d\sigma_k(x)$$

is the scalar product in the corresponding space H_k, then there exist kernels $\widetilde{K}(\xi, x)$ and $\widetilde{K}_*(\xi, x)$ such that

$$\widetilde{K}(\xi, x) \in H_1, \qquad \widetilde{K}_*(\xi, x) \in H_1 \quad (\xi \in (a_2, b_2)) \tag{1}$$

and for every $f(x) \in H_1$ we have

$$\int_{a_2}^{b_2} g(x)e_\xi(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} \overline{\widetilde{K}(\xi,x)}\,f(x)\,d\sigma_1(x), \tag{2}$$

$$\int_{a_2}^{b_2} g^*(x)e_\xi(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} \overline{\widetilde{K}_*(\xi,x)}\,f(x)\,d\sigma_1(x), \tag{3}$$

where $g(x) = R_1 f(x)$ and $g^*(x) = R_{*1} f(x)$. Here

$$e_\xi(x) = \begin{cases} 1, & x \in [0,\xi), \\ 0, & x \notin [0,\xi), \end{cases} \quad \xi > 0; \qquad e_\xi(x) = \begin{cases} -1, & x \in [\xi,0), \\ 0, & x \notin [\xi,0), \end{cases} \quad \xi < 0. \tag{4}$$

We note that $\widetilde{K}(\xi,x)$ and $\widetilde{K}_*(\xi,x)$ are related by

$$\int_{a_1}^{b_1} \overline{\widetilde{K}(\xi,x)}\,\widetilde{K}_*(\eta,x)\,d\sigma_1(x) = \int_{a_2}^{b_2} e_\xi(x)e_\eta(x)\,d\sigma_2(x). \tag{5}$$

and both are complete. In this connection we call a certain kernel $K(\xi,x)$ **complete** in H_1, if from the fact that $f(x) \in H_1$ and

$$\int_{a_1}^{b_1} K(\xi,x)\overline{f(x)}\,d\sigma_1(x) = 0$$

for all $\xi \in (a_2,b_2)$, it follows that $f(x) \equiv 0$. In the particular case of an isometric operator R, mapping all of H_1 onto all of H_2, we have $R = R_1 = R_{*1}$. In this case $\widetilde{K}(\xi,x) \equiv \widetilde{K}_*(\xi,x) \equiv K(\xi,x)$, and formulas (2), (3), and (5) hold if everywhere $K(\xi,x)$ and $\widetilde{K}_*(\xi,x)$ are replaced by $K(\xi,x)$, and one sets $g(x) = g^*(x) = Rf(x)$.

It is useful to note the following theorem.

Theorem 1. *Suppose that for all $\xi \in (a_2,b_2)$ we have $K(\xi,x) \in H_1$, where $K(\xi,x)$ is complete in H_1 and satisfies the condition*

$$\int_{a_1}^{b_1} \overline{K(\xi,x)}K(\eta,x)\,d\sigma_1(x) = \int_{a_2}^{b_2} e_\xi(x)e_\eta(x)\,d\sigma_2(x), \qquad \xi,\eta \in (a_2,b_2), \tag{6}$$

for all $\xi,\eta \in (a_2,b_2)$. Then there exist a kernel $H(\xi,x)$ complete in H_2 and an isometric operator V, mapping all of H_2 onto all of H_1, such that if one sets $g(x) = V^{-1}f(x)$, then for all $f(x) \in H_1$ and all $\xi \in (a_2,b_2)$

$$\int_{a_2}^{b_2} g(x)e_\xi(x)\,d\sigma_2(x) = \int_{a_1}^{b_1} \overline{K(\xi,x)}f(x)\,d\sigma_1(x),$$

$$\int_{a_1}^{b_1} f(x)e_\xi(x)\,d\sigma_1(x) = \int_{a_2}^{b_2} \overline{H(\xi,x)}g(x)\,d\sigma_2(x),$$

and the relations

$$\int_{a_2}^{b_2} \overline{H(\xi,x)}H(\eta,x)\,d\sigma_2(x) = \int_{a_1}^{b_1} e_\xi(x)e_\eta(x)\,d\sigma_1(x), \qquad \xi,\eta \in (a_1,b_1),$$

$$\int_{a_1}^{b_1} K(\xi,x)e_\eta(x)\,d\sigma_1(x) = \int_{a_2}^{b_2} \overline{H(\eta,x)}e_\eta(x)\,d\sigma_2(x), \qquad \xi \in (a_2,b_2), \quad \eta \in (a_1,b_1).$$

This theorem justifies the following definition. Every complete kernel $K(\xi,x)$ is called the **kernel of an isometric operator** if, for all $\xi,\eta \in (a_2,b_2)$, condition (6) is satisfied.

We shall further say that the function $\widetilde{K}(\xi,x)$, defined for all $\xi \in (a_2,b_2)$ and $x \in (a_1,b_1)$, is a B**-kernel** if there exists a function $\widetilde{K}_*(\xi,x)$, defined for the same ξ and x, such that $\widetilde{K}(\xi,x)$ and $\widetilde{K}_*(\xi,x)$ satisfy conditions (1) and (5) and, moreover, both are complete. In this connection we shall call $\widetilde{K}_*(\xi,x)$ the **kernel conjugate to** $\widetilde{K}(\xi,x)$. It is clear that the conjugate kernel is determined uniquely. It is also obvious that a kernel conjugate to a B-kernel is itself a B-kernel.

We shall further say that a B-kernel $\widetilde{K}(\xi,x)$ is **nonless**, if to every function $f(x) \in H_1$ there corresponds a certain function $g(\xi) \in H_2$ such that for all $\eta \in (a_2,b_2)$

$$\int_{a_1}^{b_1} f(x)\overline{\widetilde{K}_*(\eta,x)}\,d\sigma_1(x) = \int_{a_2}^{b_2} g(\xi)e_\eta(\xi)\,d\sigma_2(\xi). \tag{7}$$

Obviously, $g(\xi)$ is determined uniquely.

Theorem 2. *In order that a B-kernel $\widetilde{K}(\xi,x)$ be nonless, it is necessary and sufficient that there exist such a linear bounded*

operator A, mapping H_1 into itself, such that for all $\xi \in (a_2,b_2)$

$$A\widetilde{K}(\xi,x) = K(\xi,x), \tag{8}$$

where $K(\xi,x)$ is the kernel of some isometric operator mapping H_1 onto H_2.

We now introduce the notion of a Hilbert kernel. We shall say that the B-kernel $\widetilde{K}_*(\xi,x)$ is a Hilbert kernel if to every function $g(\xi) \in H_2$ there corresponds some function $f(x) \in H_1$ such that

$$\int_{a_1}^{b_1} f(x)\,\overline{\widetilde{K}(\eta,x)}\,d\sigma_1(x) = \int_{a_2}^{b_2} g(\xi)e_\eta(\xi)\,d\sigma_2(\xi). \tag{9}$$

It is then obvious that the function $f(x)$ is determined uniquely.

Theorem 3. In order that the B-kernel $\widetilde{K}_*(\xi, x)$ be a Hilbert kernel, it is necessary and sufficient that there exist a linear bounded operator C, mapping H_1 into itself, such that for all $\xi \in (a_2, b_2)$

$$\widetilde{K}_*(\xi, x) = CK(\xi, x), \tag{10}$$

where $K(\xi, x)$ is the kernel of some isometric operator.

Theorem 4. If either of the conditions (8) or (10) is satisfied, the corresponding operators A or C are invertible in the sense that from $Af = 0$ or $Cf = 0$ it follows that $f = 0$.

Theorem 5. If a certain B-kernel is Bessel, then the kernel adjoint to it is Hilbert, and conversely.

Theorem 6. In order that the B-kernel $\widetilde{K}(\xi, x)$ be Bessel, it is necessary and sufficient that there exist a positive bounded Hermitian operator T, defined on H_1, such that for all $\xi \in (a_2, b_2)$ the condition

$$\widetilde{K}_*(\xi, x) = T\widetilde{K}(\xi, x) \tag{11}$$

be satisfied.

It can be proved that, when condition (11) is satisfied, the operator is invertible, i.e., from $Tf = 0$ it follows that $f = 0$.

From this theorem there immediately follows the analogous condition for Hilbert kernels.

We shall now agree to call a certain B-kernel a **Riesz-Fischer kernel** if it is simultaneously Bessel and Hilbert. It is easy to see that if one of two adjoint B-kernels is a Riesz-Fischer kernel, then the other is as well.

Theorem 7. If $\widetilde{K}(\xi, x)$ is a Riesz-Fischer kernel, then for every $f(x) \in H_1$ there exist uniquely determined functions $g(x) \in H_2$ and $g^*(x) \in H_2$ such that the equalities

$$\int_{a_2}^{b_2} g(x) e_\xi(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} \overline{\widetilde{K}(\xi, x)}\, f(x)\, d\sigma_1(x);$$

$$\int_{a_2}^{b_2} g^*(x) e_\xi(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} \overline{\widetilde{K}_*(\xi, x)}\, f(x)\, d\sigma_1(x).$$

At the same time there exist positive constants M, m, K, k such that the inequalities

$$m\|f\|_{\sigma_1} \leqslant \|g\|_{\sigma_2} \leqslant M\|f\|_{\sigma_1}; \qquad k\|f\|_{\sigma_1} \leqslant \|g^*\|_{\sigma_2} \leqslant K\|f\|_{\sigma_1}.$$

hold.

Theorem 8. In order that the B-kernel $\widetilde{K}(\xi, x)$ be a Riesz-Fischer kernel, it is necessary and sufficient that there exist a linear bounded operator A, mapping H_1 onto all of H_1, such that $A\widetilde{K}(\xi, x) =$

$$= K(\xi, x),$$

where $K(\xi, x)$ is the kernel of some isometric operator. In this case the operator A has a bounded inverse.

Theorem 9. In order that the B-kernel $\widetilde{K}(\xi, x)$ be a Riesz–Fischer kernel, it is necessary and sufficient that there exist a bounded, positive-definite Hermitian operator T, defined on H_1, such that for all $\xi \in (a_2, b_2)$ condition (11) be satisfied.

We shall now adopt the following definition. A B-kernel $\widetilde{K}(\xi, x)$ is called a **basis kernel** or a **Riesz kernel in** H_1 if there exists a kernel $\widetilde{H}_*(\xi, x)$, complete in H_2 $(x \in (a_2, b_2))$, connected with $\widetilde{K}(\xi, x)$ by the relation

$$\int_{a_1}^{b_1} \widetilde{K}(\eta, x) e_\xi(x)\, d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}_*(\xi, x)}\, e_\eta(x)\, d\sigma_2(x),$$

so that to every function $f(x) \in H_1$ there corresponds a certain function $g(x) \in H_2$ such that

$$\int_{a_1}^{b_1} f(x) e_\xi(x)\, d\sigma_1(x) = \int_{a_2}^{b_2} \overline{\widetilde{H}_*(\xi, x)}\, g(x)\, d\sigma_2(x),$$

and moreover

$$\int_{a_2}^{b_2} g(x) e_\xi(x)\, d\sigma_2(x) = \int_{a_1}^{b_1} f(x) \overline{\widetilde{K}_*(\xi, x)}\, d\sigma_1(x).$$

This definition is justified by the following considerations. Let $\widetilde{K}_n(x)$ and $\widetilde{K}_n^*(x)$ be two complete sequences of functions in $L_2(a, b)$, forming a biorthogonal system.

Put

$$\widetilde{K}(\xi, x) = \int_0^\infty \widetilde{K}_u(x) e_\xi(x)\, d[u], \qquad x \in (a, b), \quad \xi = 1, 2, \dots,$$

$$\widetilde{H}_*(\xi, x) = \int_a^b \widetilde{K}_x^*(u) e_\xi(u)\, du, \qquad \xi \in (a,b), \quad x = 1, 2, \ldots .$$

It is easy to see that if $\{\widetilde{K}_n(x)\}$ is a basis, then for $\widetilde{K}(\xi, x)$ and $\widetilde{H}_*(\xi, x)$ all the assertions hold which were adopted in the above definition of Riesz kernels.

Theorem 10. Every B-kernel $\widetilde{K}(\xi, x)$ that is a Riesz–Fischer kernel is also a Riesz kernel. At the same time the adjoint kernel $\widetilde{K}_*(\xi, x)$ is also a Riesz kernel.

In conclusion we note that, as was established in (1) (Theorem 3), if the operators R_1 and R_{*1} form an isometric pair, then each of them generates a Riesz kernel in the sense of the definition given above, and these kernels are adjoint to one another.

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Note: Figure translations are in progress. See original paper for figures.