ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR NONLINEAR ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

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ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR NON-LINEAR ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

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In the theory of boundary-value problems for linear elliptic equations, a definite role has been played by the method of quadratic forms, or the so-called energy method, by means of which the existence theorem for the solution of the first boundary-value problem for strongly elliptic systems of equations in the space $W_2^{(m)}$, where $2m$ is the order of the system ([1]), is established rather simply. Further, one can establish the differential properties of the solution and, with the aid of S. L. Sobolev' s embedding theorems ([2]), derive conditions for the existence of a solution of the problem in the classical sense.

In the present paper an analogue of the energy method is applied to prove the existence of a solution of the first boundary-value problem for a certain class of nonlinear elliptic systems of equations.

1. For greater simplicity we give the exposition for systems of second order, having the form

$$L(u)u \equiv -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ik}(x,u)\frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{n} B_i(x,u)\frac{\partial u}{\partial x_i} + C(x,u)u = h, \quad (1)$$

where A_{ik}, B_i, C are matrices of order N; $u = (u_1, \dots, u_N)$; $h = h(x) = (h_1, \dots, h_N)$; $x \in D$; Γ is the boundary of the domain D. The extension to the case of systems of order $2m$ is obtained directly from what is set forth below. We note that one may allow $h = h(x,u)$, if $|h(x,u)| < M(x)$. The first boundary-value problem consists in finding a solution $u(x)$ of system (1) satisfying the condition

$$u|_{\Gamma} = 0. \quad (2)$$

Let $C_0^{(1)}(D)$ be the space of all continuously differentiable functions satisfying (2).

Assumptions. I. For any $w(x)$, $u(x) \in C_0^{(1)}(D)$,

$$
\begin{aligned}
(L(w)u,u) \equiv K(w;u,u) \equiv \sum \left(A_{ik}(x,w)\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_i} \right) \\
+ \sum \left(B_i(x,w)\frac{\partial u}{\partial x_i}, u \right) + (C(x,w)u,u) \qquad (3) \\
\geq c^2 \sum \left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i} \right) \equiv c^2 \|u\|_{1,2}^2,
\end{aligned}
$$

where $c^2 > 0$ and does not depend on the choice of u and w.

II. For each of the matrices A_{ik}, B_i there exists a "majorizing" invertible matrix $\widetilde{A}_{ik}$, $\widetilde{B}_i$ such that

$$
|\widetilde{A}_{ik}^{-1}(x,u)| < M, \qquad |\widetilde{B}_i^{-1}(x,u)| < M, \qquad |\widetilde{A}_{ik}^{-1*} A_{ik}^*| < M, \qquad |\widetilde{B}_i^{-1*} B_i^*| < M, \tag{4}
$$

where $|\ |$ denotes the upper bound of the moduli of the elements of the matrix with respect to all $x \in D$ and u, and the asterisk denotes passage to the adjoint matrix.

III. For some $p = 1 + \varepsilon > 1$ $(p \leqslant 2)$ and every $u \in C_0^{(1)}(D)$

$$
\sum \left\| \widetilde{A}_{ik}(x,u)\frac{\partial u}{\partial x_k} \right\|_{0,p} + \sum \left\| \widetilde{B}_i(x,u)\frac{\partial u}{\partial x_i} \right\|_{0,p} + \|C(x,u)u\|_{0,p} \leqslant f(l(u)), \tag{5}
$$

where f is a fixed monotone function, $l(u) = K(u;u,u)$, $\|\ \|_{0,p} = \|\ \|_{L_p}$. Below we shall illustrate the fulfillment of conditions I, II, III.

Conditions II, III concerning the matrices A_{ik} can be weakened, namely one may require:

II′. The matrix $A(x,u) = \|A_{ik}\|$ (of order nN) has a bounded inverse $|A^{-1}| < M$; the conditions on B_i are the same.

III′. Instead of the first term in (5), substitute

$$
\sum_i \left\| \sum_k A_{ik} \frac{\partial u}{\partial x_k} \right\|_{0,p}.
$$

A function $h(x) \in H^{(-1)}(D)$ if it can be represented in the form

$$
h = \sum_{i=1}^{n} \frac{\partial h_i}{\partial x_i},
$$

where $h_i \in L_2(D)$ (Schwartz, Lions).

Theorem 1. *If conditions I, II, III or conditions I, II′, III′ are satisfied, then for every right-hand side $h(x) \in H^{(-1)}(D)$ there exists at least one generalized solution $u(x)$ of problem (1), (2), with $u \in \overset{0}{W}{}_2^{(1)}(D)$ and having finite norms appearing on the left in (5).*

A function $u(x)$ with these finite norms and $u|_\Gamma = 0$ (in the mean) is called a generalized solution of problem (1), (2), if for every function $v \in \overset{0}{W}{}_q^{(1)}(D)$, where $q = p/(p-1)$ $(v|_\Gamma = 0)$,

$$K(u; u, v) = (h, v) = -\sum \left(h_i, \frac{\partial v}{\partial x_i}\right), \qquad \left(h = \sum \frac{\partial h_i}{\partial x_i}\right). \tag{6}$$

Proof will be carried out under conditions I, II, III. Under conditions I, II′, III′ it is entirely analogous. We shall construct a solution u of problem (1), (2) by means of the Galerkin method. For this, choose in $\overset{0}{W}{}_q^{(1)}$ a complete linearly independent system of functions $v_i(x)$, with all $v_i \in C_0^{(1)}(D)$. We shall seek the n-th approximation $\tilde{u}_n$ to the function u in the form

$$\tilde{u}_n = \sum_{i=1}^{n} C_i v_i \qquad (C_i = C_i^{(n)}),$$

where C_i $(i = 1, \dots, n)$ are found from the nonlinear system of equations

$$K(\tilde{u}_n; \tilde{u}_n, v_j) = (h, v_j) \qquad (j = 1, \dots, n). \tag{7}$$

To prove its solvability, consider the auxiliary linear system of equations

$$K(w_n; z_n, v_j) = (h, v_j) \qquad (j = 1, \dots, n), \tag{8}$$

where

$$w_n = \sum_{i=1}^{n} D_i v_i(x),$$

D_i are arbitrary fixed numbers, and

$$z_n = \sum F_i v_i(x).$$

Solving (8) for F_i, to prove the solvability of (8) we multiply its left-hand sides by F_j and sum over j from 1 to n. Then, according to (3), we obtain

$$K(w_n; z_n, z_n) \geqslant c^2 \|z_n\|_{1,2}^2, \tag{9}$$

whence it follows that the determinant of (8) is different from zero.

Denote by V_n the finite-dimensional operator which to each function w_n assigns the corresponding solution z_n of system (8);

$V_n w_n = z_n$. According to (8), we have

$$K(w_n; z_n, z_n) = (h, z_n) = -\sum \left(h_i, \frac{\partial z_n}{\partial x_i} \right) \leq M \sum \|h_i\|^2 + \frac{c^2}{2} \|z_n\|_{1,2}^2, \tag{10}$$

and, using (9), from this we derive

$$\|z_n\|^2 = \|V_n w_n\|^2 \leq M_1^2 \sum \|h_i\|^2 = M_1^2 \|h\|_{-1}^2, \tag{11}$$

where M_1 does not depend on n or on w_n. Therefore, if w_n varies in the ball $\|w_n\|_{1,2} \leq 2M_1 \|h\|_{-1}$, then the images $z_n = V_n w_n$, according to (11), lie inside the ball $\|z_n\|_{1,2} \leq M_1 \|h\|_{-1}$. Hence, by the fixed-point principle, there exists a function $\tilde{u}_n = \sum^n C_i v_i$ such that $V_n \tilde{u}_n = \tilde{u}_n$ ($\|\tilde{u}_n\|_{1,2} \leq M_1 \|h\|_{-1}$). For it (7) is fulfilled: $K(\tilde{u}_n; \tilde{u}_n, v_j) = (h, v_j)$, and, according to (10),

$$l(\tilde{u}_n) = K(\tilde{u}_n; \tilde{u}_n, \tilde{u}_n) \leq M_2 \|h\|_{-1}^2. \tag{12}$$

From (12), (5), the weak compactness of the sphere in L_p, and the embedding theorems we conclude that from $\{\tilde{u}_n\}$ one can choose a subsequence, which we shall also denote by $\{\tilde{u}_n\}$, such that: 1) $\tilde{A}_{ik}(x, \tilde{u}_n)\, \partial \tilde{u}_n / \partial x_k$ converges weakly in L_p to an element which we denote by $\omega_{ik}(x)$, and $\tilde{B}_i(x, \tilde{u}_n)\, \partial \tilde{u}_n / \partial x_i$ to $\gamma_i(x) \in L_p$; 2) $\partial \tilde{u}_n / \partial x_i$ converges weakly to $\partial u / \partial x_i$ in L_2; 3) $\tilde{u}_n(x)$ converges almost everywhere in D to $u(x)$.

For any function $\psi(x)$ continuous in $D + \Gamma$ (or $\psi(x) \in L_q(D)$),

$$\lim_{n \to \infty} \left(\tilde{A}_{ik}(x, \tilde{u}_n)\, \partial \tilde{u}_n / \partial x_k, \psi \right) = \left(\tilde{A}_{ik}(x, u)\, \partial u / \partial x_k, \psi \right). \tag{13}$$

Indeed, we have:

$$(\partial \tilde{u}_n / \partial x_k, \psi) \to (\partial u / \partial x_k, \psi).$$

On the other hand,

$$(\partial \tilde{u}_n / \partial x_k, \psi) = (\tilde{A}_{ik}(x, \tilde{u}_n)\, \partial \tilde{u}_n / \partial x_k, \tilde{A}_{ik}^{-1*}(x, \tilde{u}_n) \psi) \to (\omega_{ik}, \tilde{A}_{ik}^{-1*}(x, u) \psi) = (\tilde{A}_{ik}^{-1} \omega_{ik}, \psi).$$

Consequently, almost everywhere in D,

$$\omega_{ik}(x) = \tilde{A}_{ik}(x, u)\, \partial u / \partial x_k.$$

Next we have:

$$
\begin{aligned}
(\tilde{A}_{ik}(x,\tilde{u}_n)\,\partial\tilde{u}_n/\partial x_k,\psi) &= (\tilde{A}_{ik}(x,\tilde{u}_n)\,\partial\tilde{u}_n/\partial x_k, \tilde{A}_{ik}^{-1*}(x,\tilde{u}_n)\tilde{A}_{ik}^{*}(x,u)\psi)\\
&\to (\tilde{A}_{ik}(x,u)\,\partial u/\partial x_k, \tilde{A}_{ik}^{-1*}(x,u)\tilde{A}_{ik}^{*}(x,u)\psi)\\
&= (\tilde{A}_{ik}(x,u)\,\partial u/\partial x_k,\psi).
\end{aligned}
$$

Here we used the fact that $\tilde{u}_n(x)$ converges almost everywhere in D to $u(x)$ and that the matrices $\tilde{A}_{ik}^{-1*}\tilde{A}_{ik}^{*}$ are bounded (see (4)). Analogously one verifies that

$$
(B_i(x,\tilde{u}_n)\,\partial\tilde{u}_n/\partial x_i,\psi) \to \left(B_i(x,u)\frac{\partial u}{\partial x_i},\psi\right) \tag{14}
$$

$$
(C(x,\tilde{u}_n)\tilde{u}_n,\psi) \to (C(x,u)u,\psi).
$$

It follows from (13) and (14) that the function $u = \lim_{n\to\infty}\tilde{u}_n$ is a solution of the posed problem. Indeed, passing to the limit in (7) as $n\to\infty$, we obtain relation (6), in which v is replaced by v_j. Since by linear combinations of the v_j one can approximate any function $v \in \mathring{W}_q^{(1)}(D)$, it follows that u satisfies (6), as was required to prove.

2. Example A. Consider the case of one elliptic equation of the form (1), and, for brevity, suppose that $B_i \equiv 0$, $C \equiv 0$. Let the following conditions be fulfilled:

1) $A_{ii}(x,u) = a_{ii}(x,u)(1+|u|^{2\alpha_i})$, where $0 < c_1^2 \le a_{ii}(x,u) \le C_1^2$, and α_i are arbitrary fixed positive numbers.

2) $\sum A_{ik}(x,u)\xi_i\xi_k \ge c^2 \sum A_{ii}(x,u)\xi_i^2$, where c^2 does not depend on ξ_i, x, u.

Then all the conditions of Theorem 1 are fulfilled, and problem (1), (2) has at least one solution for any right-hand side $h \in H^{(-1)}$.

Indeed, the estimate holds

$$
(A_{ii}(x,u)\,\partial u/\partial x_i,\ \partial u/\partial x_i) \ge c_1^2\left((1+|u|^{2\alpha_i})\,\partial u/\partial x_i,\ \partial u/\partial x_i\right) \ge
$$

$$
\ge c_2^2\left(|u|^{\alpha_i+1},\ |u|^{\alpha_i+1}\right). \tag{15}
$$

Hence we easily derive that, for some $p = 1+\varepsilon$, we have $\|A_{ii}(x,u)\,\partial u/\partial x_i\|_{0,p} \le f(l(u))$, where f is a certain power function, $l(u) = K(u;u,u)$. Further, from 2) it follows that $A_{ik}^2(x,u) \le A_{ii}(x,u)A_{kk}(x,u) \le C(1+|u|^{2\alpha_i})(1+|u|^{2\alpha_k})$. Hence, and from (15), we obtain $\|A_{ik}(x,u)\,\partial u/\partial x_k\|_{0,p} \le f(l(u))$. If now we take $\widetilde{A}_{ii}(x,u) = A_{ii}(x,u)$, $\widetilde{A}_{ik}(x,u) = [A_{ii}(x,u)A_{kk}(x,u)]^{1/2}$, then, as is easily verified, all the conditions of the theorem are satisfied, and problem (1), (2) is solvable in the sense indicated above.

Example B. Let us now consider an example of the first boundary-value problem for a nonlinear equation of the 4th order, $L_4 u = h$, whose bilinear form has, for simplicity, the form

$$(L_4 u, v) \equiv K(u; u, v) \equiv \int_D \sum_{i=1}^{n} \left(a_i(x) + b_i(x) u^{2\alpha_i} \left(\frac{\partial u}{\partial x_i} \right)^{2l_i} \right) \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 v}{\partial x_i^2}\, dx,$$

$$u\big|_\Gamma = v\big|_\Gamma = \partial u/\partial n\big|_\Gamma = \partial v/\partial n\big|_\Gamma = 0,$$

α_i, l_i are arbitrary natural numbers (for simplicity), $a_i(x) > 0$, $b_i(x) > 0$ for $x \in D + \Gamma$.

The estimate holds

$$\left\| \left(a_i + b_i u^{2\alpha_i} (\partial u/\partial x_i)^{2l_i} \right) \partial^2 u/\partial x_i^2 \right\|_{0,p} \le f(l(u)) \tag{16}$$

($l(u) = K(u; u, u)$), where f is a certain power function, $p = 1 + \varepsilon$. In deriving this estimate the following is used.

Lemma. For any $u \in C_0^{(2)}(D + \Gamma)$

$$\cdots \le C_\alpha \int_D u^{2\alpha_i + 2} \left(\frac{\partial u}{\partial x_i} \right)^{2l_i} dx \le C_{\alpha+1} \int_D u^{2\alpha_i} \left(\frac{\partial u}{\partial x_i} \right)^{2l_i + 2} dx \le$$

$$\le C_{\alpha+2} \int_D u^{2\alpha_i} \left(\frac{\partial u}{\partial x_i} \right)^{2l_i} \left(\frac{\partial^2 u}{\partial x_i^2} \right)^2 dx.$$

For the proof it suffices to integrate over D both sides of the identity

$$\left(u^{2\alpha_i + 1} (\partial u/\partial x_i)^{2l_i + 1} \right)'_{x_i} =$$

$$= (2\alpha_i + 1) u^{2\alpha_i} (\partial u/\partial x_i)^{2l_i + 2} + (2l_i + 1) u^{2\alpha_i + 1} (\partial u/\partial x_i)^{2l_i} \partial^2 u/\partial x_i^2;$$

to use the fact that the integral of the left-hand side is equal to zero, and then successively apply the Cauchy-Schwarz inequality. There is a theorem analogous to Theorem 1 for equations of the 4th order. Its conditions—the fulfillment of estimate (16) and the condition of positive definiteness of $K(w; u, u)$—are satisfied in the example under consideration. Consequently, there exists at least one solution $u(x)$ satisfying the relation $K(u; u, v) = (h, v)$ for any $h \in H^{(-2)}$.

3. The generalization of Theorem 1 to the case of systems of order $2m$ and to the corresponding operator equations is obvious.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.