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Abstract

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I. E. Burmistrovich

EMBEDDING AN ADDITIVE CATEGORY IN A CATEGORY WITH DIRECT PRODUCTS

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In this note the category of matrices over a given additive category K is introduced, and it is proved that it is the minimal extension of the category K in which direct products exist for arbitrary finite sets of objects*. From this there follows, in particular, a theorem on the endomorphism ring of an abelian group decomposed into a direct sum of a finite number of subgroups (see ([4])).

Definition 1. We shall call a category **additive** if addition is defined in each set $H(a, b)$, with respect to which it is an abelian semigroup with zero, and this addition is connected with multiplication by the distributive laws.

Definition 2. An object g of an additive category is called the **direct product** of the objects $a_i,\ i = 1, \dots, n$, if mappings $\pi_i : g \to a_i$, called **projections**, and $\sigma_i : a_i \to g$, called **embeddings**, are given, and moreover

$$\sigma_i \pi_j = \begin{cases} \varepsilon_{a_i}, & \text{for } i = j, \\ \omega, & \text{for } i \neq j; \end{cases} \tag{1}$$

$$\sum_{i=1}^{n} \pi_i \sigma_i = \varepsilon_g. \tag{2}$$

We shall denote the direct product by

$$g = a_1 \times a_2 \times \dots \times a_n \quad (\pi_i, \sigma_i).$$

Definition 3. Let an additive category $\overline{K}$ be given, and in it a full** subcategory K. We shall say that $\overline{K}$ is a **completion** of K if the direct product of any two objects of $\overline{K}$ exists and every object of $\overline{K}$ is the direct product of a finite number of objects of K.

Definition 4. The **skeleton** of a category K is the full subcategory $S(K)$ generated by objects chosen one from each class of equivalent objects. Categories with isomorphic skeletons are called **coextensive**.

For what follows we need to construct the **category of matrices** K^M over the additive category K. As objects of K^M we take all possible finite ordered sets of objects from K, not necessarily distinct. Let $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ be two such sets. By a mapping $h : A \to B$

* For definitions and notation from category theory see ([1–3]).

** That is, a subcategory containing, together with any two objects a, b, the entire set $H(a, b)$.

we shall mean any matrix in which, at the intersection of the i-th row and the j-th column, there stands a mapping $\alpha_{ij} : a_i \to b_j,\ \alpha_{ij} \in K;\ i = 1, \dots, m;\ j = 1, \dots, n$.

Let us call the pair of collections $(a_1, \dots, a_m;\ b_1, \dots, b_n)$ the type of the matrix h.

Since the category K is additive, these matrices can be added and multiplied according to the rules usual for addition and multiplication of rectangular matrices.

It is easy to see that K^M will be an additive category; moreover, if in the category K all $H(a, b)$ were abelian groups under addition, then the same is true in K^M. For an object $A = (a_1, \dots, a_m)$ of the category K^M, the identity mapping is the matrix

$$\varepsilon_A = \begin{pmatrix} \varepsilon_{a_1} & 0 & & \\ & \varepsilon_{a_2} & & \\ & & \ddots & \\ 0 & & & \varepsilon_{a_m} \end{pmatrix}.$$

It is obvious that $(K^M)^M \simeq K^M$.

A special case of the category K^M is the collection of all rectangular matrices over an associative ring with identity—the latter is an additive category with one object.

Lemma. *The object $C = (a_1, \dots, a_m, b_1, \dots, b_n)$ of the category K^M is the direct product of the objects $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$.*

Indeed, the projections are the matrices

$$\pi_1 = \begin{pmatrix} \varepsilon_{a_1} & 0 \\ & \ddots \\ 0 & \varepsilon_{a_m} \\ & 0 \end{pmatrix} \qquad (m + n \text{ rows},\ m \text{ columns}),$$

$$\pi_2 = \begin{pmatrix} 0 & \\ \varepsilon_{b_1} & 0 \\ & \ddots \\ 0 & \varepsilon_{b_n} \end{pmatrix} \qquad (m+n \text{ rows}, \ n \text{ columns}),$$

and the embeddings are the matrices

$$\sigma_1 = \begin{pmatrix} \varepsilon_{a_1} & 0 & \\ & \ddots & 0 \\ 0 & \varepsilon_{a_m} & \end{pmatrix} \qquad (m \text{ rows}, \ m+n \text{ columns}),$$

$$\sigma_2 = \begin{pmatrix} \varepsilon_{b_1} & 0 & \\ 0 & \ddots & \\ & 0 & \varepsilon_{b_n} \end{pmatrix} \qquad (n \text{ rows}, \ m+n \text{ columns}).$$

The matrices for the projections and embeddings are obtained, respectively, from the matrix ε_C if one "cuts" it in the vertical or, respectively, in the horizontal direction.

Properties (1) and (2) are obvious.

Theorem. *The category K^M is a completion of the additive category K. This completion is unique up to coessentiality.*

The first assertion of the theorem follows immediately from the preceding lemma, since every object $A = (a_1, \ldots, a_m)$ of the category K^M will be the direct product of the objects $a_1, \ldots, a_m$ of the category K.

Let now an additive category K and some completion $\overline{K}$ of it be given. It is necessary to show that the skeletons $S(K^M)$ and $S(\overline{K})$ of the categories K^M and $\overline{K}$ are isomorphic.

For this we shall need the following

Lemma. *There exists an additive covariant functor*

$$F: \ K^M \longrightarrow S(\overline{K}),$$

which maps K^M onto the whole category $S(\overline{K})$ and has the property that the full inverse image of every object of $S(\overline{K})$ is a class of equivalent objects in K^M.

We define the functor F as follows. To an object $A = (a_1, \ldots, a_m)$ of K^M we assign the object $F(A) = a = a_1 \times \cdots \times a_m$ (π_i, σ_i) of $S(\overline{K})$. Here it is not assumed that $a_i, \pi_i, \sigma_i \in S(\overline{K})$, but the object $a = a_1 \times \cdots \times a_m$ in $S(\overline{K})$, by assumption, exists and is unique. To a matrix (α_{ij}) of type $(a_1, \ldots, a_m; \ b_1, \ldots, b_n)$ we assign the mapping

$$F((\alpha_{ij})) = \sum_{i,j} \pi_i \alpha_{ij} \sigma'_j : \ a \longrightarrow b,$$

where

$$a = a_1 \times \cdots \times a_m \quad (\pi_i, \sigma_i),$$

$$b = b_1 \times \cdots \times b_n \quad (\pi'_j, \sigma'_j).$$

It is easy to see that this functor satisfies the requirements of the lemma. Therefore the functor

$$F_1 : \ S(K^M) \longrightarrow S(\overline{K}),$$

induced by the functor F, gives an isomorphic mapping of the skeleton $S(K^M)$ onto the skeleton $S(\overline{K})$.

Moscow State University
named after M. V. Lomonosov

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[4] A. G. Kurosh, *Group Theory*, Moscow, 1953, p. 140.

Note: Figure translations are in progress. See original paper for figures.