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Abstract

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MATHEMATICS

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ON AN EXPLICIT EXPRESSION OF THE CONNECTION FORMS OF A QUASISYMMETRIC SPACE THROUGH THE VALUES OF THE CURVATURE AND TORSION TENSORS AT A CERTAIN POINT

(Presented by Academician S. L. Sobolev on 20 I 1960)

A **quasisymmetric space of affine connection** is a space of affine connection with covariantly constant curvature and torsion tensors. P. K. Rashevskii showed that every quasisymmetric space is a homogeneous space (1), and every homogeneous space G/H with transformation group G and stationary group H, and such that the Cartan metric of the group G on H is nondegenerate, is quasisymmetric (2). The connection in an affine connection space A_n in a moving frame is specified by smooth linear differential forms $\omega^i(d)$ and $\omega^i_j(d)$, depending on the coordinates of the space $u^1, u^2, \dots, u^n$, on the secondary parameters of the moving frame $u^{n+1}, \dots, u^L$, and on the differentials du^i, du^α $(i = 1, 2, \dots, n;\ \alpha = n+1, \dots, L)$; the forms ω^i are linearly independent forms into which only the differentials du^i enter.

Introduce the notation:

$$[\delta\omega(d)] = \delta\omega(d) - d\omega(\delta),$$

$$[\omega_1(\delta)\omega_2(d)] = \omega_1(\delta)\omega_2(d) - \omega_1(d)\omega_2(\delta). \tag{1}$$

In the affine connection space A_n the structure equations hold:

$$[\delta\omega^i(d)] + [\omega^i_k(\delta)\omega^k(d)] = S^i_{pq}\omega^p(\delta)\omega^q(d); \tag{2}$$

$$[\delta\omega^i_j(d)] + [\omega^i_k(\delta)\omega^k_j(d)] = -R^i_{\cdot j,pq}\omega^p(\delta)\omega^q(d), \tag{3}$$

where $R^i_{\cdot j,pq}$ is the curvature tensor; S^i_{pq} is the torsion tensor of the space A_n in the moving frame; δ and d are symbols of differentials of infinitesimal linearly independent displacements, moreover such that $\delta df - d\delta f = 0$ for any at least twice continuously differentiable $f = f(u^1, \dots, u^n, u^{n+1}, \dots, u^L)$. As shown in (1), the structure equations of the group of motions G of the quasisymmetric space $A_n = G/H$ have the form

$$[\delta\psi^i(d)] + d^i_{k\gamma}[\psi^\gamma(\delta)\psi^k(d)] = S^i_{pq}\psi^p(\delta)\psi^q(d); \tag{4}$$

$$[\delta\psi^\gamma(d)] + c^\gamma_{\beta\alpha}\psi^\alpha(\delta)\psi^\beta(d) = -b^\gamma_{pq}\psi^p(\delta)\psi^q(d), \tag{5}$$

$$p, q, i, k, i = 1, 2, \dots, n; \qquad \gamma, \alpha, \beta = n+1, \dots, r.$$

In addition,

$$\omega^i_j(d) = d^i_{j\alpha}\psi^\alpha(d), \qquad \omega^i(d) = \psi^i(d). \tag{6}$$

The quantities $S^i_{pq}, d^i_{k\gamma}, -b^\gamma_{pq}, c^\gamma_{\beta\alpha}$ are the structural constants of the group G. Comparing (4), (5), (6) with (1) and (2), we obtain

$$R^i_{\cdot j,pq} = d^i_{j\gamma} b^\gamma_{pq}. \tag{7}$$

If the structure equations of a certain Lie group have the form

$$[d\psi^I(d)] = B^I_{JK}\psi^J(\delta)\psi^K(d); \qquad I, J, K = 1, 2, \dots, r, \tag{8}$$

then, as is well known from the general theory of Lie groups (3), the coefficients $\psi^I_K(u)$ of the basic forms $\psi^I(d) = \psi^I_K(u)\,du^K$ in canonical coordinates u^K $(K = 1, 2, \dots, r)$ have, in matrix notation, the form

$$\bar\psi(u) = \sum_{m=0}^{\infty} \frac{2}{(m+1)!}\,\bar B^m, \qquad \bar\psi(u) = \|\psi^I_K(u)\|, \qquad \bar B = \|B^I_{KJ}u^J\|. \tag{9}$$

We pass to the canonical frame (one at each point) by means of the condition

$$u^\alpha = 0, \qquad \alpha = n+1,\ n+2, \dots, r. \tag{10}$$

Since in our case $u^\alpha = 0$, in order to find $\psi^i(d)$, $\psi^\alpha(d)$ it is enough to find ψ^i_j, ψ^α_i. From (9) it is seen that for this it is necessary to consider $(\bar B^m)^i_j$, $(\bar B^m)^\alpha_j$. We shall use the obvious equalities

$$(\bar B^{m+1})^i_j = (\bar B)^i_k(\bar B^m)^k_j + (\bar B)^i_\alpha(\bar B^m)^\alpha_j, \qquad (\bar B^{m+1})^\alpha_j = (\bar B)^\alpha_k(\bar B^m)^k_j + (\bar B)^\alpha_\beta(\bar B^m)^\beta_j. \tag{11}$$

For the structures (4) and (5), owing to condition (10), we have

$$\underset{0}{B}{}^k_j = S^k_{jl}u^l, \qquad \underset{0}{B}{}^\alpha_j = -b^\alpha_{jl}u^l, \qquad \underset{0}{B}{}^i_\alpha = a^i_{\alpha l}u^l, \qquad \underset{0}{B}{}^\alpha_\beta = 0, \tag{12}$$

where $\underset{0}{B}{}^I_J$ is the value of the matrix B^I_J when $u^\alpha = 0$. Therefore (11) has the form

$$\Big(\underset{0}{\bar B}{}^{m+1}\Big)^i_j = \underset{0}{B}{}^i_k \Big(\underset{0}{\bar B}{}^m\Big)^k_j + \underset{0}{B}{}^i_\beta \Big(\underset{0}{\bar B}{}^m\Big)^\beta_j, \qquad \Big(\underset{0}{\bar B}{}^{m+1}\Big)^\alpha_j = \underset{0}{B}{}^\alpha_k \Big(\underset{0}{\bar B}{}^m\Big)^k_j. \tag{13}$$

Eliminating $\Big(\underset{0}{\bar B}{}^m\Big)^\beta_j$ from (13), we obtain

$$\Big(\underset{0}{\bar B}{}^{m+2}\Big)^i_j = S^i_k \Big(\underset{0}{\bar B}{}^{m+1}\Big)^k_j + N^i_k \Big(\underset{0}{\bar B}{}^m\Big)^k_j, \qquad N^i_k = R^i{}_{.l,ks}u^l u^s, \qquad S^i_k = \underset{0}{B}{}^i_k. \tag{14}$$

The expressions (14) make sense starting with $m = 0$. Using also (6), (12), (14), we obtain

$$\omega^i_{jk} = a^i_{j\alpha}\psi^\alpha_k = -R^i{}_{.j,lq}u^q \sum_{m=0}^{\infty} \frac{1}{(m+2)!}\Big(\underset{0}{\bar B}{}^m\Big)^l_k. \tag{15}$$

Thus,

$$\omega^i = \sum_{m=0}^{\infty} \frac{1}{(m+1)!}\, b^i_m, \qquad \omega^i_j = -R^i{}_{.j,lq}u^q\varphi^l,$$

$$\varphi^l = \sum_{m=0}^{\infty} \frac{1}{(m+2)!}\, b^l_m, \qquad b^l_m = \Big(\underset{0}{\bar B}{}^m\Big)^l_k\, du^k, \tag{16}$$

where, with the help of (14) and (12), we have

$$b^i_{m+2} = S^i_k b^k_{m+1} + N^i_k b^k_m, \qquad b^k_0 = du^k, \qquad b^k_1 = S^k_j\, du^j, \qquad i, j, k = 1, 2, \dots, n. \tag{17}$$

To solve equation (17), let us pass to the vector space of $2n$ variables ξ^a $(a = 1, 2, \dots, 2n)$. Consider vectors $\underset{m}{\xi}{}^a$ such that $\underset{m}{\xi}{}^i = \underset{m}{b}{}^i$, $\underset{m}{\xi}{}^{n+i} = \underset{m+1}{b}{}^i$ $(i = 1, 2, \dots, n)$, and a matrix D^a_b such that $D^i_j = 0$, $D^{n+i}_{n+j} = S^i_j$, $D^{n+i}_j = N^i_j$, $D^i_{n+j} = \delta^i_j$, $i = 1, 2, \dots, n$. Then equation (17) can be rewritten in the form

$$\underset{m+1}{\xi}{}^a = D^a_b \underset{m}{\xi}{}^b, \qquad \underset{0}{\xi}{}^i = du^i, \qquad \underset{0}{\xi}{}^{n+i} = S^i_j du^j. \tag{18}$$

From (18) it follows that

$$\underset{s}{\xi}^a = (\overline{D}^{s+1})^a_b \xi^b, \qquad \xi^i = 0, \qquad \xi^{n+i} = du^i, \qquad i = 1, 2, \dots, n. \tag{19}$$

Introduce a form Ω^a such that

$$\Omega^i = \varphi^i, \qquad \Omega^{n+i} = \omega^i. \tag{20}$$

Then, obviously, the expansion

$$\Omega^a = \xi^a + \sum_{m=0}^{\infty} \frac{1}{(m+2)!} \underset{m}{\xi}^a \tag{21}$$

holds.

Let us note that if a function $f(\overline{A})$ is representable by a power series $\sum_{m=1}^{\infty} a_m \overline{A}^m$, then division by $\overline{A}$, $\frac{1}{\overline{A}} f(\overline{A})$, is possible even if $\overline{A}$ is a degenerate matrix. By $\frac{1}{\overline{A}} f(\overline{A})$ we shall mean the series $\sum_{m=1}^{\infty} a_m \overline{A}^{m-1}$.

Taking (19) into account, we obtain further

$$\overline{\Omega} = \begin{pmatrix} \overline{\varphi} \\ \overline{\omega} \end{pmatrix} = \left(\frac{e^{\overline{D}} - \overline{E}}{\overline{D}} \right) \begin{pmatrix} \overline{0} \\ \overline{du} \end{pmatrix}, \qquad \overline{\Omega}_1 = \frac{1}{2} \left(\frac{e^{\overline{D}} - \overline{E}}{\overline{D}} + \frac{e^{(\overline{BDB})} - \overline{E}}{(\overline{BDB})} \right) \begin{pmatrix} \overline{0} \\ \overline{du} \end{pmatrix} = \begin{pmatrix} \overline{0} \\ \overline{\omega} \end{pmatrix}$$

$$\overline{\Omega}_2 = \frac{1}{2} \left(\frac{e^{\overline{D}} - \overline{E}}{\overline{D}} - \frac{e^{(\overline{BDB})} - \overline{E}}{(\overline{BDB})} \right) \begin{pmatrix} \overline{0} \\ \overline{du} \end{pmatrix} = \begin{pmatrix} \overline{\varphi} \\ \overline{0} \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} -\overline{I} & \overline{0} \\ \overline{0} & \overline{I} \end{pmatrix}, \qquad \overline{E} = \begin{pmatrix} \overline{I} & \overline{0} \\ \overline{0} & \overline{I} \end{pmatrix} \tag{22,}$$

$$\overline{D} = \begin{pmatrix} \overline{0} & \overline{I} \\ \overline{N} & \overline{S} \end{pmatrix}, \qquad \overline{B}\,\overline{D}\,\overline{B} = -\begin{pmatrix} \overline{0} & \overline{I} \\ \overline{N} & -\overline{S} \end{pmatrix} = -\overline{\mathfrak{D}}, \qquad \overline{S} = \| S^i_{jq} u^q \|, \qquad \overline{I} = \| \delta^i_j \|,$$

$$\overline{N} = \| R^i_{\cdot j,ls} u^l u^s \|, \qquad \overline{\varphi} = \begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{pmatrix}, \qquad \overline{\omega} = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}, \qquad \overline{du} = \begin{pmatrix} du^1 \\ \vdots \\ du^n \end{pmatrix}, \qquad \omega^i_j = -R^i_{\cdot j,lq} u^q \varphi^l.$$

If the quasi-symmetric space A_n under consideration admits the existence of a covariantly constant nondegenerate positive-definite tensor field $g_{ij}(u)$, then, as can be shown, the basic forms ω^i can be chosen so that

$$ds^2 = \sum_{i=1}^{n} (\omega^i)^2 = (\overline{0}\ \overline{\omega}) \cdot \begin{pmatrix} \overline{0} \\ \overline{\omega} \end{pmatrix} = \overline{\Omega}_1^* \cdot \overline{\Omega}_1. \tag{23}$$

We shall make use of this choice of basis. Then

$$ds^2 = \frac{1}{4}\,\overline{\xi}^* \left(\frac{e^{\overline{D}^*} - \overline{E}}{\overline{D}^*} - \frac{e^{-\overline{\mathfrak{D}}^*} - \overline{E}}{\overline{\mathfrak{D}}^*} \right) \left(\frac{e^{\overline{D}} - \overline{E}}{\overline{D}} - \frac{e^{-\overline{\mathfrak{D}}} - \overline{E}}{\overline{\mathfrak{D}}} \right) \overline{\xi}, \qquad \overline{\xi} = \begin{pmatrix} \overline{0} \\ du \end{pmatrix}. \tag{24}$$

Expressions analogous to (24), but somewhat more complicated, can also be given in the case of an indefinite metric.

If, in particular, the space is symmetric, then

$$\bar{\bar{S}} = 0, \qquad \overline{D} = \begin{pmatrix} \bar{\bar{0}} & \bar{\bar{I}} \\ \bar{\bar{N}} & \bar{\bar{0}} \end{pmatrix} = \overline{\mathcal{D}}, \qquad \bar{\bar{N}}^* = \bar{\bar{N}}, \qquad \overline{D}^2 = \begin{pmatrix} \bar{\bar{N}} & \bar{\bar{0}} \\ \bar{\bar{0}} & \bar{\bar{N}} \end{pmatrix},$$

and thus:

$$\begin{aligned} \bar{\omega} &= \sum_{m=0}^{\infty} \frac{\bar{\bar{N}}^m \, d\bar{u}}{(2m+1)!} = \frac{\operatorname{sh}\sqrt{\bar{\bar{N}}}}{\sqrt{\bar{\bar{N}}}}\, d\bar{u}, \\ \bar{\varphi} &= \sum_{m=0}^{\infty} \frac{\bar{\bar{N}}^m \, d\bar{u}}{(2m+2)!} = \frac{\operatorname{ch}\sqrt{\bar{\bar{N}}} - \bar{\bar{I}}}{\bar{\bar{N}}}\, d\bar{u}. \end{aligned} \tag{25}$$

Using (24), we obtain

$$ds^2 = \bar{\omega}^* \bar{\omega} = d\bar{u}^* \left(\frac{\operatorname{sh}^2 \sqrt{\bar{\bar{N}}}}{\bar{\bar{N}}} \right) d\bar{u} = d\bar{u}^* \left(\frac{\operatorname{ch}\left(2\sqrt{\bar{\bar{N}}}\right) - \bar{\bar{I}}}{2\bar{\bar{N}}} \right) d\bar{u}. \tag{26}$$

Formulas equivalent to (26) were first obtained by P. A. Shirokov[4], but by an entirely different method.

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Note: Figure translations are in progress. See original paper for figures.

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