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Abstract

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MATHEMATICS

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ERROR ESTIMATE FOR THE GRID METHOD FOR THE TWO-DIMENSIONAL NEUMANN PROBLEM

(Presented by Academician S. L. Sobolev on 16 II 1960)

In works ($^{6-8}$) the author proposed a method for the approximate solution of the two-dimensional Neumann problem using various approximations of the boundary condition for the adjoint function; for quadratic interpolation an error estimate is given in (^{8}), whereas in the case of linear interpolation or transfer of the boundary values of the adjoint function to the boundary nodes of the grid, the method of error estimation indicated in (^{8}) is not suitable, since in this case the boundary condition for the approximate solution of the Neumann problem is satisfied in the weak sense and, consequently, locally (with respect to the grid step) is satisfied poorly because of the presence of the normal derivative in the boundary condition; for example, when boundary values are transferred, the local approximation is of order $O(1)$. Therefore, in such a situation, taking into account that inside the domain the solution is smooth, it is natural to give an error estimate only in an interior subdomain. For this purpose we shall apply an estimation method based on embedding theorems and on an inequality derived in (^{4}).

Consider the difference Dirichlet problem for the equation

$$\Delta_h u_n = 0 \tag{1}$$

in the domain $\Omega(x_1, x_2)$, whose boundary S consists of a finite number of rectifiable curves; the function u_n in (1) is defined at the nodes of the grid domain $\Omega_h + S_h$.

In addition to the notation of works ($^{6-8}$), we introduce the following: on the set of boundary points S_h define an analogue of the normal derivative:

$$\frac{\Delta u}{\Delta n} = \frac{1}{h}(ku_0 - u_1 - \ldots - u_k),$$

where $u_0, u_1, \dots, u_k$ are the values of the function u_n at the given boundary node and at k neighboring interior nodes (^{6}); denote by $D_h^\beta u_n$ the divided difference of the function u_n of order β.

If now $\eta = U - u_n$ is the error of the solution of equation (1), and U is the exact solution, then

$$\Delta_h \eta = \psi_1, \tag{2}$$

and, as is known, inside the domain Ω, $D_h^\beta \psi_1 = O(h^2)$.

In work (^{4}), for functions specified on a grid in the domain Ω, the inequality

$$h^2 \sum_{\Omega^{**}} (w_{x_1}^2 + w_{x_2}^2) \leq K(w) = c_1 h^2 \sum_{\Omega^*} w^2 + c_2 h^2 \sum_{\Omega^*} (\Delta_h w)^2, \tag{3}$$

was proved, where the domain Ω^{**} is entirely contained inside the domain Ω^*, which in turn lies inside Ω.

From the validity of inequality (3) there follows

Corollary. If $h^2 \sum_{\Omega^*} \eta^2 \leq c_3 h^{2\alpha}$ $(0 < \alpha \leq 2)$, then $\max_{\Omega'} |D_h^\beta \eta| \leq c'_\beta h^\alpha$, where the domain Ω' is contained entirely in Ω^{**}.

Indeed, without loss of generality we may assume that $\Omega^*, \Omega^{**}, \Omega'$ are concentric squares with boundaries Q^*, Q^{**}, Q', coinciding with the lines forming the mesh; then from the embedding theorems (^{9}, ^{5}) and inequality (3) it follows that

$$h \sum_{\Omega^{**}} \left(\eta_{x_1}^2 + \eta_{x_2}^2 \right) \leq c_4 K(\eta), \tag{4}$$

$$h \sum_{Q^{**}} \eta^2 \leq c_5 K(\eta), \tag{5}$$

and, consequently, expressing η in Ω' through the difference Green' s function G_h for the square Ω^{**} (^{4}, ^{10}) and through the values of η on Q^{**}, we obtain

$$|\eta| \leq \left| h^2 \sum_{\Omega^{**}} G \psi_1 \right| + \left| h \sum_{Q^{**}} \frac{\Delta G_h}{\Delta n} \eta \right| \leq c_6 \sqrt{K(\eta)} + c_7 h^2. \tag{6}$$

The required inequality will be obtained if, successively in the same way, we obtain for $D_h^\beta \eta$, $\beta = 1, 2, \dots$, inequalities (4), (5), (6), in whose left-hand sides $D_h^\beta \eta$ stands instead of η (^{5}), and if into the right-hand side of formula (6), written for $D_h^\beta(\eta)$, we substitute the estimate for η.

The known inequalities (4), (5), (6) for η and $D_h^\beta \eta$ will be needed to estimate the error of the difference Neumann problem. Indeed, let $\zeta = V - v_h$ be the error in the solution of the Neumann problem, and let η be the error of the conjugate Dirichlet problem, obtained in solving the equations considered in (^{6}, ^{7}); then

$$\zeta_{x_1} = \eta_{x_2} + \psi_3, \qquad \zeta_{x_2} = -\eta_{x_1} + \psi_4, \tag{7}$$

where $D_h^\beta \psi_i = O(h^2)$, $i = 3, 4$, in Ω^{**}. Expressing η_{x_1}, η_{x_2} from (7) and substituting into (4), we obtain

$$h^2 \sum_{\Omega^{**}1} \left(\zeta_{x_1}^2 + \zeta_{x_2}^2 \right) \le c_4 K(\eta) + c_8 h^4, \tag{8}$$

where the square $\Omega_1^{**} \subset \Omega^{**}$ has sides Q_1^{**} coinciding with straight lines forming the mesh for the function v_h. Choosing the constant in the definition of the function ζ so that $h^2 \sum_{\Omega^{**}1} \zeta = 0$, we obtain from (8) that

$$h \sum_{Q^{**}1} \zeta^2 \le c_9 K(\eta) + c_{10} h^4,$$

and, consequently, in the square $\Omega_1' \subset \Omega_1^{**}$ the inequality obtained in the same way as inequality (6) will be valid:

$$\max_{\Omega_1'} |\zeta| \le c_{11} \sqrt{K(\eta)} + c_{12} h^2. \tag{9}$$

Similarly we obtain that

$$\max_{\Omega_1'} |D_h^\beta \zeta| \le c_\beta'' \sqrt{K(\eta)} + c_\beta''' h^2. \tag{10}$$

Knowing the estimates for the quantity $\eta^{(1,2)}$, from inequalities (9), (10) we obtain estimates for the quantity ζ, i.e., if $|\eta| = O(h^\alpha)$, $0 < \alpha \leqslant 2$, then also

$$\max_{\Omega_1'} |D_h^\beta \zeta| = O(h^\alpha), \quad \beta = 0, 1, 2, \ldots$$

Thus, we see that inside the domain Ω the error in the solution of the Neumann problem is of the same order as the error in the solution by the grid method of the adjoint Dirichlet problem; in particular, in the case of sufficient smoothness of the exact solution, when very simple boundary operators are used (^{6}), where the value $\Delta v_n / \Delta n$, locally approximating the normal derivative, is prescribed on the boundary only in the directions of the coordinate axes and the bisectors of the coordinate angles, accuracy of order h is attained inside the domain, and in the case of linear interpolation—of order h^2.

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* **Correction.** In the author' s article ([8]), devoted to estimating the error of the Neumann problem, the following corrections must be made:

1) In the definition of the function $F(r)$ the factor h^2 was omitted, and in the formula for $z(r)$ the logarithm in the second term must have the form $\ln \frac{r}{\tau}$.
2) The investigation of the quantity ψ_1 near the boundary of the domain can be carried out as follows. Let $U \in H(2, A, \gamma)$ ([3]); then, using Taylor formulas of the type

$$U(h) = U(0) + hU'_x(0) + \frac{h^2}{2} U''_{xx}(\theta h),$$

we obtain that

$$\psi_1 = \Delta_h U = \sum_{i=1}^{2} U''_{x_i x_i}(x_i + \theta_i h) = O(h^\gamma),$$

where $|\theta_i| \leqslant 1$.

If $U \in H(3, A, \gamma)$, then, taking into account one more term in the Taylor formulas, we obtain

$$\psi_1 = \Delta_h U = \frac{h}{3!} \sum_{i=1}^{2} \left[U'''_{x_i x_i x_i}(x_i + \theta_{i1} h) - U'''_{x_i x_i x_i}(x_i - \theta_{i2} h) \right] = O(h^{1+\gamma});$$

where $0 \leqslant \theta_{ik} \leqslant 1$. Such a representation of the function ψ_1 makes it possible, taking into account the results of (1), to investigate the behavior of ψ_τ near the boundary of the domain and to obtain for it the estimate in the paper (8).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.