

# ON A THEOREM OF THE QUALITATIVE THEORY OF PARABOLIC EQUATIONS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON A THEOREM OF THE QUALITATIVE THEORY OF PARABOLIC EQUATIONS**

*(Presented by Academician I. G. Petrovskii, July 20, 1959)*

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} = a(t, x) \frac{\partial u}{\partial t} + b(t, x) \frac{\partial u}{\partial x} + c(t, x)u, \quad (1)$$

given in the closed rectangular domain  $\bar{G} = (0 \leq t \leq T \leq 1; \delta \leq x \leq 1 - \delta, 1/2 > \delta > 0)$  of the plane  $(t, x)$ .

The coefficients of equation (1) satisfy in  $\bar{G}$  the following conditions:

- 1) all coefficients are bounded in modulus by one;
- 2)  $a(t, x) \geq a_0 > 0, c(t, x) \geq 0$ ;
- 3)  $a(t, x), b(t, x), c(t, x) \in C^{(2)}$  with respect to  $x$  and are bounded in modulus together with all required derivatives;
- 4) the coefficient  $a(t, x)$  has a derivative with respect to  $t$ ;
- 5)  $|\partial a(t, x)/\partial t| \leq 1, |\partial b(t, x)/\partial x| \leq 1$ .

The solution  $u(t, x)$  of equation (1) will be assumed to be twice continuously differentiable in  $\bar{G}$ , to belong to  $C^{(4)}$  with respect to  $x$ , and such that  $|u(t, x)| < 1$ .

Let some positive number  $\xi < \frac{1 - 2\delta}{4}$  be given.

Introduce the following notation:

- $\Sigma = (\delta \leq x \leq 1 - \delta, t = T)$ ;
- $\xi_1 = (\delta \leq x < \delta + \xi, t = T)$ ,
- $\xi_2 = (\delta + \xi \leq x < \delta + 2\xi, t = T)$ ,
- $\xi_3 = (1 - \delta - 2\xi < x \leq 1 - \delta - \xi, t = T)$ ,
- $\xi_4 = (1 - \delta - \xi < x \leq 1 - \delta, t = T)$ ;
- $\Sigma^* = \Sigma \setminus \xi_1 \setminus \xi_2 \setminus \xi_3 \setminus \xi_4$ ;
- $\Pi_T = (0 < t < T), \Pi_2 = (T/2 < t < T)$ ,
- $\Pi^{**} = (t_1 < t < T) (T/2 \leq t_1 < T)$ ;
- $\gamma = T - t_1$ ;
- $\Delta_1 = (\delta < x < \delta + \xi, T/2 < t < T)$ ,
- $\Delta_2 = (1 - \delta - \xi < x < 1 - \delta, T/2 < t < T)$ ,

$$(G \cap \Pi_2) \setminus \Delta_1 \setminus \Delta_2 = G_{2\Delta};$$

$$(1 - 2\delta)T = \sigma.$$

Put, as was done in <sup>(1)</sup>,

$$G_+ = \{(t, x) \in G, u(t, x) > 0\}, \quad G_- = \{(t, x) \in G, u(t, x) < 0\}.$$

We shall call essential those components of the sets  $G_+$  and  $G_-$  which have limit points on both sides of the strip  $\Pi_T$ .

Suppose that the solution  $u(t, x)$  has in the domain  $G$   $N$  essential components  $g_i$  ( $i = 1, \dots, N$ ). Suppose that  $N_1$  essential components have limit points on  $\Sigma^*$ .

The proof of the following lemma is, in idea, analogous to the proof of Theorem 2.6.1 of <sup>(1)</sup>.

**Lemma.** There exist absolute constants  $M_1$  and  $M_2$  such that, if

$$N_1 \geq \frac{4M_1^{1/2}M_2^{1/2}\sigma}{\gamma^{3/2}t_1^{3/2}},$$

then for every level line  $l$  connecting both sides of the strip  $\Pi^{**}$ , the inequality

$$|u|_l < 2^{-\frac{t_1^3 N_1^2}{4M_1 \sigma^2}}$$

holds.

( $M_1 = 80^2$ ,  $M_2 > 640^2$ —absolute constants, respectively, of Theorem 2.3.1 and Lemma 2.5.1 of [1]).

In what follows we shall assume that

$$N_1 \geq \frac{32M_1^{1/2}M_2^{1/2}(1 - 2\delta)}{T^2}. \quad (2)$$

1°. In the case when there are essential components having limit points on  $\xi_2$  (on  $\xi_1$  and  $\xi_2$ ) and on  $\xi_3$  (on  $\xi_3$  and  $\xi_4$ ), the dependence between  $\max_{\Sigma^*} |u(T, x)|$  and the number  $N_1$  is established by a direct application of Theorem 2.6.1 of [1].

2°. Consider the case when among the essential components there are some that have limit points on  $\xi_2$  (on  $\xi_1$  and  $\xi_2$ ), but there are no essential components having limit points on  $\xi_3$  and  $\xi_4$ , or, conversely, there are essential components having limit points on  $\xi_3$  (on  $\xi_3$  and  $\xi_4$ ), and there are no essential components having limit points on  $\xi_1$  and  $\xi_2$ , as well as the case when  $N_1 = N$ .

Let  $\max_{\Sigma^*} |u(T, x)| = 2\varepsilon_0$ . Denote  $S_1 = \{(t, x) \in \Sigma^*, |u(t, x)| = 2\varepsilon_0\}$ ,  $S_2 = \{(t, x) \in \Sigma^*, |u| = \varepsilon_0\}$ . Obviously, the sets  $S_1$  and  $S_2$  are nonempty.

The following subcases are possible:

2<sub>1</sub><sup>o</sup>. The set  $S_2$  contains at least one point  $(T, x')$  such that the level line  $l'$  passing through it connects both sides of the strip  $\Pi_2$  (the set  $S_1$  may have no such point).

Put  $t_1 = T/2$  (then, consequently,  $\gamma = T/2$ ). Using the lemma, we find that

$$\max_{\Sigma^*} |u(T, x)| = 2\varepsilon_0 < 2 \cdot 2^{-\frac{1}{32} \frac{TN^2}{(1-2\delta)^2 M_1}}. \quad (3)$$

2<sub>2</sub><sup>o</sup>. The level lines passing through points of the sets  $S_1$  and  $S_2$  have no common points with the straight line  $t = T/2$ .

Let the point  $(T, x_2) \in S_1$ . Without loss of generality, we may assume that  $u(T, x_2)$  is positive. Let  $(T, x_3) \in S_2$  belong to the same component  $G_+$  to which  $(T, x_2)$  belongs. Denote by  $l_{2\varepsilon_0}$  and  $l_{\varepsilon_0}$  the level lines passing, respectively, through the points  $(T, x_2)$  and  $(T, x_3)$ . The lines  $l_{2\varepsilon_0}$  and  $l_{\varepsilon_0}$  exit either through the left or through the right boundary of the domain  $G$ . The two cases are symmetric; therefore in what follows we shall assume that the lines  $l_{2\varepsilon_0}$  and  $l_{\varepsilon_0}$  go to the left and that  $(T, x_2)$  is situated to the left of  $(T, x_3)$ .

Obviously,

$$1 - 2\delta - 3\xi > r_0 \geq \xi, \quad \text{where } r_0 = x_2 - \delta - \xi.$$

On the basis of (2), for the domain  $\bar{G}$  one can indicate a constant  $\widetilde{M}_2 \geq 2$ , depending only on the coefficients  $a(t, x)$ ,  $b(t, x)$ ,  $c(t, x)$  and their derivatives, such that

$$\left| \frac{\partial^2 u}{\partial x^2} \right| < \frac{\widetilde{M}_2 (1 - 2\delta)^2 \max_{\bar{G}} |u|}{t \left[ \left( \frac{1-2\delta}{2} \right)^2 - \left( \frac{1-2x}{2} \right)^2 \right]^2}.$$

For the domain  $\bar{G}_{2\Delta}$  we have

$$\left| \frac{\partial^2 u}{\partial x^2} \right| < \frac{2\widetilde{M}_2 (1 - 2\delta)^2 \max_{\bar{G}} |u|}{T\xi^2 (1 - 2\delta - \xi)^2}. \quad (4)$$

The set of values of the function  $u(T, x)$  on  $\Sigma^*$  is bounded by constants  $\underline{m}$  and  $\overline{m}$ . By the Kronrod-Landis theorem [3], for almost all  $\tau$  of the interval  $[\underline{m}, \overline{m}]$  the corresponding level sets  $L_\tau = \{(t, x) \in G, u(t, x) = \tau\}$

do not have points at which  $\text{grad } u = 0$ . Denote by  $\mathfrak{M}$  the set of such  $\tau^* \in [\underline{m}, \bar{m}]$  for which at no point  $(t^*, x^*) \in L_{\tau^*}$  does  $\text{grad } u(t^*, x^*)$  vanish.

Obviously,  $\mu_1 \mathfrak{M} = \bar{m} - \underline{m}$ . Suppose first that  $2\varepsilon_0$  and  $\varepsilon_0$  belong to  $\mathfrak{M}$ . Then the lines  $l_{2\varepsilon_0}$  and  $l_{\varepsilon_0}$  are smooth curves. Denote by  $\mathcal{T}$  the set of points lying between  $l_{2\varepsilon_0}$  and  $l_{\varepsilon_0}$ .

Put also  $U = (x_1 = \delta + \xi < x < x_2 = \delta + \xi + r_0)$ ,  $\Omega = \mathcal{T} \cap U$ . Let the straight line  $t = t_2$  pass through the point belonging to  $l_{\varepsilon_0}$  and having the smallest  $t$ -coordinate in  $\overline{G_{2\Delta}}$ .

We have

$$0 = \iint_{\Omega} a(t, x) \frac{\partial u}{\partial t} dt dx + \iint_{\Omega} b(t, x) \frac{\partial u}{\partial x} dt dx + \iint_{\Omega} c(t, x) u dt dx - \iint_{\Omega} \frac{\partial^2 u}{\partial x^2} dt dx = I_1 + I_2 + I_3 + I_4.$$

Since

$$\begin{aligned} I_1 &> \varepsilon_0 a_0 r_0 - 3\varepsilon_0 r_0 (T - t_2) - r_0 (T - t_2), \\ I_2 &> (-3\varepsilon_0 r_0 - 6\varepsilon_0 - 2 - r_0) (T - t_2), \\ I_3 &> -r_0 (T - t_2), \quad I_4 > -\frac{2\widetilde{M}_2(1 - 2\delta)^2}{T\xi^2(1 - 2\delta - \xi)^2} (T - t_2) r_0^*, \end{aligned}$$

we obtain that

$$T - t_2 > \frac{\varepsilon_0 a_0 r_0}{6\varepsilon_0 r_0 + 2 + 3r_0 + 6\varepsilon_0 + \frac{2\widetilde{M}_2(1 - 2\delta)^2}{T\xi^2(1 - 2\delta - \xi)^2} r_0}.$$

Obviously,

$$\frac{\varepsilon_0 a_0 r_0}{6\varepsilon_0 r_0 + 2 + 3r_0 + 6\varepsilon_0 + \frac{2\widetilde{M}_2(1 - 2\delta)^2}{T\xi^2(1 - 2\delta - \xi)^2}} > \frac{\varepsilon_0 r_0 a_0}{3\widetilde{M}_2(1 - 2\delta)^2} \frac{1}{T\xi^2(1 - 2\delta - \xi)^2}.$$

Put

$$\gamma_0 = \frac{\varepsilon_0 r_0 a_0}{3\widetilde{M}_2(1 - 2\delta)^2} \frac{1}{T\xi^2(1 - 2\delta - \xi)^2}$$

(obviously,  $\gamma_0 < T/96$ ) and  $t_1^0 = T - \gamma_0$ ; then

$$T - t_2 > \gamma_0.$$

We shall show that

$$N_1 < \frac{4M_1^{1/2}M_2^{1/2}(1-2\delta)T}{\gamma_0^{3/2}t_1^{03/2}}. \quad (5)$$

Suppose the contrary. Then, by the lemma,

$$\frac{3\widetilde{M}_2(1-2\delta)^2\gamma_0}{T\xi^2(1-2\delta-\xi)^2a_0r_0} = \varepsilon_0 < 2^{-\frac{t_1^{03}N_1^2}{4M_1\xi^2}},$$

whence it follows that

$$\log_2 \frac{T\xi^2(1-2\delta-\xi)^2a_0r_0}{3\widetilde{M}_2(1-2\delta)^2\gamma_0} > \frac{4M_2}{\gamma_0^3},$$

which is impossible. Inequality (5) is proved.

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\* The integrals  $I_1$  and  $I_2$  are estimated with the aid of Green's formula. In estimating the integral  $I_4$ , estimate (4) is used.

It is obvious that

$$N_1^{2/3} < \frac{3 \cdot 2^{4/3} M_1^{1/3} \widetilde{M}_2 (1-2\delta)^{3/8} T^{1/3} M_2^{1/3}}{t_1^0 \varepsilon_0 a_0 \xi^2 T (1-2\delta-\xi)^2 r_0} < \frac{2^{16/3} M_1^{1/3} M_2^{1/3} \widetilde{M}_2 (1-2\delta)^{2/3}}{\max_{\Sigma^*} |u(T, x)| a_0 T^{4/3} \xi^2 r_0},$$

whence

$$\max_{\Sigma^*} |u(T, x)| < \frac{2^{16/3} M_1^{1/3} M_2^{1/3} \widetilde{M}_2 (1-2\delta)^{2/3}}{N_1^{2/3} \xi^3 a_0 T^{1/3}}. \quad (6)$$

If at least one of the values  $2\varepsilon_0$  and  $\varepsilon_0$  does not belong to  $\mathfrak{M}$ , we choose  $2\varepsilon'_0$  and  $\varepsilon''_0 \in \mathfrak{M}$ , arbitrarily close to  $2\varepsilon_0$  and  $\varepsilon_0$ , respectively, so that  $2\varepsilon'_0 - \varepsilon''_0 = \varepsilon'''_0 \gg \varepsilon_0$ . Carrying out arguments and calculations analogous to those carried out for the case when  $2\varepsilon_0$  and  $\varepsilon_0 \in \mathfrak{M}$ , we obtain that

$$\max_{\Sigma^*} |u(T, x)| = 2\varepsilon_0 \leq 2\varepsilon''_0 < \frac{2^{16/3} M_1^{1/3} M_2^{1/3} \widetilde{M}_2 (1-2\delta)^{2/3}}{N_1^{2/3} \xi^3 a_0 T^{1/3}}. \quad (6')$$

Thus, in the case  $\mathfrak{D}_1^0$  we obtain inequality (3), and in the case  $\mathfrak{D}_2^0$ , inequality (6). Since estimate (6) is the coarsest, it is suitable for all cases.

**Theorem.** *Let, in the rectangular domain  $G$ , the number  $N_1$  of essential components having limit points on  $\Sigma^*$  satisfy condition (2). Then*

$$\max_{\Sigma^*} |u(T, x)| < MN_1^{-2/3} \xi^{-3},$$

where

$$M = \frac{2^{16/3} M_1^{1/3} M_2^{1/3} \widetilde{M}_2 (1 - 2\delta)^{2/3}}{a_0 T^{1/3}}$$

depends only on the domain  $G$  and the coefficients of equation (1), and the positive number

$$\xi < \frac{1 - 2\delta}{4}.$$

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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