MATHEMATICS

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Fig. 1

Figure 1: Fig. 1

Abstract

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ENERGY INEQUALITIES FOR CERTAIN CLASSES OF EQUATIONS OF MIXED TYPE

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In the paper (^{1}) a method was given for proving energy inequalities for equations in partial derivatives, applicable in a number of cases; the existence of energy inequalities ensured solvability in the weak sense of the corresponding boundary-value problems. In particular, in (^{1}) one class of equations of mixed type was considered. We now extend these investigations to more general equations of mixed type and to more general domains. Below we use the terminology introduced in (^{1}). We note that the method applied below may be regarded as a development of the *abc*-method for proving uniqueness theorems for Chaplygin' s equation.

1°. Let G be a finite domain in the plane (x_1, x_2), contained in the strip $-h \leq x_2 \leq H$ and bounded by a piecewise-smooth curve Γ; G intersects the axis Ox_1. We shall study in G an equation of mixed type generalizing Chaplygin' s equation. Namely, consider the differential expression with real continuous coefficients in $G \cup \Gamma$

$$\mathcal{L}[u] = \sum_{j,k=1}^{3} D_j(a_{jk}(x) D_k u) +$$

$$+ \sum_{j=1}^{2} a_j(x) D_j u + a(x) u \tag{1}$$

$$(D_j = d/dx_j, \quad j = 1, 2).$$

Fig. 1

Assume that $\mathcal{L}$ is elliptic in the domain $G_e = G \cap \{x_2 > 0\}$ in the sense that, for some $\varepsilon > 0$,

$$\sum a_{jk}(x)\xi_j\xi_k \geq \varepsilon|\xi_2|^2;$$

moreover, let

$$2a(x) - \sum D_j a_j(x) \leq 0 \quad (x \in G_e);$$

the coefficients $a_{jk}(x)$ and $a_j(x)$ are continuously differentiable in $G \cup \Gamma$. Let, as x passes into the hyperbolic domain $G_g = G \cap \{x_2 < 0\}$, the coefficients of $\mathcal{L}$ continuously pass into the coefficients of the hyperbolic expression $k(x_2)D_1^2 + D_2^2$. Here $k(x_2)$ is continuous in $[-h, 0]$ and continuously differentiable in $[-h, 0)$, with

$$k(x_2) < 0, \quad k'(x_2) > 0 \quad \text{in } [-h, 0),$$

$$\lim_{x_2 \to 0} (k(x_2)/k'(x_2)) = 0$$

and $(k/k')'$ summable in $[-h, 0]$. Assume that the domain G has the more special form indicated in Fig. 1. Here Γ_v is an arbitrary piecewise-smooth curve; $\Gamma_{1,\mathrm{l}}, \Gamma_{1,\mathrm{p}}, \ldots, \Gamma_{N,\mathrm{l}}, \Gamma_{N,\mathrm{p}}$ are arcs of characteristics (in the figure $N = 3$); γ_l and γ_p are two arcs having, respectively, the equations $x_2 = \alpha_l(x_1)$, $x_2 = \alpha_p(x_1)$, where

$$\alpha_l \geq 0, \quad \alpha_l' < (-k)^{-1/2}, \quad \alpha_p \leq 0, \quad |\alpha_p'| < (-k)^{-1/2}.$$

Some of the arcs Γ, γ may be absent.

Theorem 1. Consider the boundary conditions (bc),

$$u\big|_{\Gamma_v \cup \gamma_l \cup \gamma_p \cup \Gamma_{1,\mathrm{l}} \cup \cdots \cup \Gamma_{N,\mathrm{l}}} = 0, \qquad u\big|_{\Gamma_{1,\mathrm{p}} \cup \cdots \cup \Gamma_{N,\mathrm{p}}} \quad \text{is removed}$$

and the adjoint boundary conditions (gr)$^+$

$$v\big|_{\Gamma_{\mathrm{B}} \cup \gamma_\ell \cup \gamma_{\mathrm{n}} \cup \Gamma_{1,\mathrm{n}} \cup \cdots \cup \Gamma_{N,\mathrm{n}}} = 0, \qquad v\big|_{\Gamma_{1,\ell} \cup \cdots \cup \Gamma_{N,\ell}} \text{ is removed.}$$

Introduce two positive norms:

$$\|u\|_+^2 = \int_G |u|^2\,dx + \int_{G_E} \sum_{j,k=1}^{2} a_{jk}(x) D_k u\,\overline{D_j u}\,dx + \int_{G_\Gamma} \left|\sqrt{-k}\,D_1 u - D_2 u\right|^2\,dx$$

and $\|v\|^2_{+,*}$, which is defined in the same way as $\|u\|^2_+$, except that in the integral over G_Γ the last minus sign is replaced by a plus sign.

Then the energy inequalities

$$\|\mathcal{L}[u]\|_0 \geq c\|u\|_+, \qquad \|\mathcal{L}[v]\|_0 \geq c\|v\|_{+,*}, \quad (u \in W_2^2(\mathrm{gr}),\ v \in W_2^2(\mathrm{gr})^+,\ c > 0)\,; \tag{2}$$

hold. Here $\|f\|_0$ is the norm in $L_2(G)$; $W_2^2(\mathrm{gr})$, $W_2^2(\mathrm{gr})^+$ are the sets of functions from $W_2^2(G)$ satisfying, respectively, the boundary conditions (gr) and (gr)$^+$.

It is also assumed that the coefficient $k(x_2)$ additionally satisfies one of the following conditions: 1) $2(k/k')'+1 \geq \delta > 0$ ($x_2 \in [-h, 0]$); 2) the coefficient $k(x_2)$ is three times continuously differentiable on $[-h, 0)$, $k''(x_2)$ is bounded near 0, and the set of those x_2 for which $2(k/k')' + 1 \leq 0$ consists of a finite number of intervals in $[-h, 0)$, and on each such interval $k''' \leq 0$ (or, more generally, $15k'^3 - 18kk'k'' + 4k^2k''' < 0$); 3) the coefficient $k(x_2)$ is twice continuously differentiable on $[-h, 0)$ and $k''(x_2)$ is bounded near 0; then necessarily, near zero, $2(k/k')' + 1 \geq 0$; let $[-d, 0)$ be the maximal interval on which this inequality is fulfilled.

The assertion of the theorem is valid if $h > d$, but $h-d$ is sufficiently small (the smallness of $h - d$ depends on k, ε, and H; $h - d$ can be estimated).

Recall (see (1)) that the second of inequalities (2) ensures the existence of a weak solution of the boundary-value problem

$$\mathcal{L}[u] = f, \quad u_{\Gamma_{\mathrm{E}}\cup\gamma_\ell\cup\gamma_{\mathrm{n}}\cup\Gamma_{1,\ell}\cup\dots\cup\Gamma_{N,\ell}} = 0, \qquad u\Big|_{\Gamma_{1,\mathrm{n}}\cup\dots\cup\Gamma_{N,\mathrm{n}}} \text{ is removed}, \tag{3}$$

i.e., the existence of such a function $u \in L_2$ for which, for every $v \in W_2^2(\mathrm{gr})^+$, the equality $(u, \mathcal{L}[v])_0 = (f, v)_0$ holds. The right-hand side f is some generalized function, namely an element of the space with negative norm constructed from the positive norm $\|v\|_{+,*}$; in any case, any function from L_2 may be taken as f. The first of inequalities (2) ensures uniqueness of a smooth solution of problem (3).

The boundary-value problem for the Chaplygin equation $k(x_2)D_1^2u + D_2^2u = f$ was studied (mainly by classical methods) in the works (2–5), as well as by other authors (see the survey (6)). Considerations close to ours are found in (7, 8); however, even for the Chaplygin equation, the nature of our restrictions and estimates (2) is different. We note that conditions 1) and 2) are, respectively, the conditions of F. I. Frankl (2) and Protter (5), under which they proved the uniqueness theorem for the Chaplygin equation.

2°. Let us briefly outline the proof of the first of inequalities (2); the second is proved analogously. Denote by $G_{1,\Gamma}$ the domain in the half-plane $x_2 < 0$

bounded by the arcs AC_1, γ_ℓ, $\Gamma_{1,\ell}$, and $\Gamma_{1,\mathrm{n}}$. For an arbitrary smooth function $u(x)$ $(x \in G_{1,\Gamma})$ the equality* holds

$$I_{G_{1,\Gamma}} = 2\,\mathrm{Re} \int_{G_{1,\Gamma}} \mathcal{L}[u] \left\{ q\tau \left(\sqrt{-k}\, D_1 u - D_2 u \right) + \gamma u \right\} dx =$$

$$= \int_{G_{1,\Gamma}} ((q\tau)' - 2\gamma) \left| \sqrt{-k}\, D_1 u - D_2 u \right|^2 dx + \int_{G_{1,\Gamma}} |u|^2 \gamma'' \, dx + I_{\Gamma_{1,\Gamma}},$$

$$\begin{aligned} I_{\Gamma_{1,r}} = 2\,\mathrm{Re} \int_{\Gamma_{1,r}} & D_1 u D_2 \bar{u} (\sqrt{-k}\nu_2 - k\nu_1) q\gamma \, dx + \\ & + \int_{\Gamma_{1,r}} (-k|D_1 u|^2 + |D_2 u|^2)(-\sqrt{-k}\,\nu_1 - \nu_2) q\gamma \, dx + \\ & + 2\,\mathrm{Re} \int_{\Gamma_{1,r}} (kD_1 u \cdot \nu_1 + D_2 u \cdot \nu_2) u\gamma \, dx - \int_{\Gamma_{1,r}} |u|^2 \gamma' \nu_2 \, dx, \end{aligned} \tag{4}$$

where $\Gamma_{1,r}$ is the boundary of $G_{1,r}$; $\nu(x)$ is the unit vector of the outward normal; $q = -4k/k'$; $\gamma(x_2)$ is an arbitrary real smooth function on $[-h, 0]$. This equality is proved as follows: consider, for example, the integral

$$F = \int_{G_{1,r}} \mathcal{L}[u] q\gamma \sqrt{-k} D_1 \bar{u} \, dx$$

and transfer, by integration by parts, $\mathcal{L}$ onto $\bar{u}$ and D_1 onto u, while in the transfer avoiding the appearance of third derivatives. On the right one obtains the term $-\bar{F}$, whence we find $2\,\mathrm{Re}\, F$. We proceed analogously with the two other integrals.

Assume that the function γ on $[-h, 0]$ satisfies the requirements

$$\gamma(0) < 0, \qquad \gamma'(0) < \frac{2\varepsilon}{H}(-\gamma(0)), \qquad \gamma'' \geqslant 0, \qquad q\gamma' - \gamma \geqslant 0, \qquad (q\gamma)' - 2\gamma \geqslant \delta > 0, \tag{5}$$

and that $u(x)$ satisfies the boundary conditions (bc). Examining in (4) the signs of the integrals and then discarding the nonnegative terms, we find

$$I_{G_{1,r}} \geqslant \delta \int_{G_{1,r}} |\sqrt{-k} D_1 u - D_2 u|^2 \, dx + 2\gamma(0)\,\mathrm{Re} \int_{AC_1} D_2 u \cdot \bar{u} \, dx - \gamma'(0) \int_{AC_1} |u|^2 \, dx. \tag{6}$$

Introduce analogous integrals $I_{G_{2,r}}, \dots, I_{G_{N,r}}$ and write for them estimates of the form (6). Adding these inequalities, we find an estimate of type (6), in which $G_{1,r}$ is replaced by G_r, and AC_1 by AB.

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* An integral similar to $I_{G_{1,\Gamma}}$ was also considered by Protter (^{5}).

Consider the elliptic subdomain G_E. Integrating by parts with allowance for $u|_{\Gamma_B}=0$, and then estimating, we find

$$2\gamma(0)\,\mathrm{Re}\int_{G_E}\mathcal{L}[u]\bar{u}\,dx \geqslant -2\gamma(0)\sum\int_{G_E} a_{jk}D_{kuD}j\bar{u}\,dx - 2\gamma(0)\,\mathrm{Re}\int_{AB} D_2u\cdot\bar{u}\,dx. \tag{7}$$

Adding (6) (for I_{G_r}) and (7), we obtain ($\chi_E(x)$ is the characteristic function of E)

$$2\,\mathrm{Re}\int_{G}\mathcal{L}[u](\chi_{G_r}\{q\gamma(\sqrt{-k}D_1u - D_2u)+\gamma u\}+\gamma(0)\chi_{G_E}\bar{u})\,dx \geqslant \tag{8}$$

$$\geqslant -2\gamma(0)\sum\int_{G_E} a_{jk}D_{kuD}j\bar{u}\,dx + \delta\int_{G_r}|\sqrt{-k}D_1u - D_2u|^2\,dx - \gamma'(0)\int_{AB}|u|^2\,dx.$$

Taking into account that the integral of the square of the modulus of a derivative is estimated from below by the integral of the square of the modulus of the function plus the square of the modulus of the function at one of the endpoints of the arc of integration, the right-hand side of (8) can be estimated from below by the expression $\eta\|u\|_+^2$. Since the left-hand side is, evidently, estimated from above by the expression $C\|\mathcal{L}[u]\|_0\|u\|_+$, as a result we obtain $C\|\mathcal{L}[u]\|_0\|u\|_+ \geqslant \eta\|u\|_+^2$, whence, after cancellation by $\|u\|_+$, we arrive at the first of inequalities (2).

Thus, to complete the proof, we need to be able to choose once and for all a continuously differentiable function $\gamma(x_2)$, ($x_2 \in [-h,0]$), having a piecewise-continuous second derivative (such smoothness

sufficient for the preceding arguments) and satisfying inequalities (5). It can be shown that in all three cases appearing in the formulation of the theorem this can be done (in the first case it suffices to set $\gamma(x_2)\equiv -1$). This completes the proof.

3°. The gas-dynamic problem of the outflow of a supersonic jet from an unbounded vessel leads, in the plane case, to the consideration of a boundary-value problem for Chaplygin' s equation in a domain G, whose elliptic part is unbounded. When the proofs of item 2° are carried over to this case, there arise, as is seen from (8), difficulties connected with the need to estimate the integral

$$\int_{G_\mathrm{e}}(|k|\,|D_1u|^2 + |D_2u|^2)\,dx$$

from below in terms of $\|u\|_0^2$, which is impossible, since $\mathcal{L}$ is not invertible in the unbounded domain G_e. However, $\mathcal{L}$ can be inverted in G_e if, as the norm $\|u\|_0$, one takes not the $L_2(G)$-norm, but an L_2-norm with weight (cf. (9)). Along these lines one can prove:

Theorem 2. Let G be an unbounded domain whose hyperbolic part is the same as before, and whose elliptic part is bounded by two arcs $\Gamma_{\mathrm{v,l}}$ and $\Gamma_{\mathrm{v,p}}$ going to ∞ (see Fig. 1); let $\mathcal{L}$ be the differential Chaplygin expression with continuous coefficient $k(x_2)$ $(-h \leq x_2 < \infty)$, satisfying for $x_2 \leq 0$ the preceding general conditions and condition 1), and for $x_2 > 0$ the requirement $k(x_2) > 0$. Consider the boundary conditions of Theorem 1, in which $\Gamma_{\mathrm{v}} = \Gamma_{\mathrm{v,l}} \cup \Gamma_{\mathrm{v,p}}$. Let $\varepsilon \in (0,1)$ be fixed; introduce the zero and positive norms

$$\|v\|_0^2 = \int_G |v|^2 \frac{dx}{(1+|x_2|)^{2-\varepsilon}},$$

$$\|v\|_{+,*}^2 = \|v\|_0^2 + \int_{G_{\mathrm{e}}} (|k|\,|D_1 v|^2 + |D_2 v|^2)(1+|x_2|)^{\varepsilon}\,dx + \int_{G_{\mathrm{r}}} |\sqrt{-k}\,D_1 v + D_2 v|^2\,dx.$$

For functions v, finite at infinity, locally belonging to W_2^2 and satisfying the boundary conditions $(\mathrm{gr})^+$, the energy inequality

$$\|((1+|x_2|)^2 \mathcal{L}[v])\|_0 \geq c\,\|v\|_{+,*}$$

is valid, ensuring the existence of a weak solution of the boundary-value problem

$$\mathcal{L}[u] = f, \qquad u \in (\mathrm{gr}), \qquad \int_G |u|^2 \frac{dx}{(1+|x_2|)^{2+\varepsilon}} < \infty, \tag{9}$$

i.e., the existence of a locally square-summable function u for which the integral (9) converges and such that $(u, \mathcal{L}[v])_{L_2} = (f, v)_L$ for every v of the indicated form. As f one may take any function for which

$$\int_G |f|^2 (1+|x_2|)^{2-\varepsilon}\,dx$$

converges, or a generalized function obtained by completing f in the norm

$$\|f\|_{-,*} = \sup_v ((f,v)_{L_2} / \|v\|_{+,*}).$$

Let, in particular, $\Gamma_{\mathrm{v,l}}$ and $\Gamma_{\mathrm{v,p}}$ be vertical half-lines, and let $k(x_2) \equiv 1$, starting from sufficiently large $x_2 > 0$. Then the boundary conditions (9) under consideration are correct in the sense that the uniqueness theorem for smooth solutions is valid: every solution u of the equation $\mathcal{L}[u] = 0$, locally belonging to W_2^2, satisfying the boundary condition (gr), and such that the integral (9) converges, is identically zero.

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