Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Vol. 133, No. 6

MATHEMATICS

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COMMUTATIVE FREE AND ANTICOMMUTATIVE FREE PRODUCTS OF ALGEBRAS

(Presented by Academician A. I. Maltsev on 14 IV 1960)

1. Let

$$p_i(x_1, \dots, x_{n_i}) \qquad (i = 1, 2, \dots)$$

be some collection of elements of the free nonassociative algebra over a field Ω of characteristic zero. All polynomials p_i are assumed to be multilinear and homogeneous of degree greater than 1.

If $A_\alpha,\ \alpha \in S$, are arbitrary algebras over the same field Ω, then by

$$\bar{G} = \prod_{\alpha \in S}^{*} A_\alpha$$

we denote the nonassociative free product of the algebras A_α, constructed for some choice of bases L_α of the algebras A_α. Let I be the set of all elements of $\bar{G}$ of the form

$$\sum a b_1 b_2 \dots b_k p_i(\mathfrak{A}_1, \dots, \mathfrak{A}_{n_i}) c_1 c_2 \dots c_l, \tag{1}$$

where $\alpha \in \Omega$, $b, \mathfrak{A}, c$ with indices are regular words in the sense of Kurosh (see (1), p. 245) of the algebra $\bar{G}$, and, moreover, if the words $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_{n_i}$ all have length 1, then at least two of them lie in different algebras A_α. In the product $b_1 b_2 \dots b_k p_i(\mathfrak{A}_1, \dots, \mathfrak{A}_{n_i}) c_1 c_2 \dots c_l$, parentheses are placed in some manner. It is not difficult to verify that I is a two-sided ideal in the algebra $\bar{G}$.

Definition 1. The factor algebra $G = \bar{G}/I$ will be called the **reduced free product** of the algebras $A_\alpha,\ \alpha \in S$, defined by the identical relations

$$p_i(x_1, \dots, x_{n_i}) = 0 \qquad (i = 1, 2, \dots), \tag{P}$$

or, more briefly, the P**-free product** of the algebras A_α, and will be denoted by

$$G = \prod_\alpha^\nabla A_\alpha = A_1 \nabla A_2 \nabla \cdots .$$

It turns out that the algebra G contains subalgebras $\mathfrak{A}_\alpha,\ \alpha \in S$, isomorphic respectively to the algebras A_α, and is generated by them. It is not difficult to show that the definition of the algebra G does not depend on the choice of the bases L_α of the algebras A_α.

We give a second definition of the P-free product of algebras.

Definition 2. Let each algebra $A_\alpha,\ \alpha \in S$, be given by a system of generators M_α and a system of defining relations Φ_α, with the sets $M_\alpha,\ \alpha \in S$, pairwise disjoint. Denote by M the union of all the sets M_α: $M = \bigcup_{\alpha \in S} M_\alpha$, by Φ the union of all the systems Φ_α: $\Phi = \bigcup_{\alpha \in S} \Phi_\alpha$, and by Ψ the system of all possible defining

relations of the form

$$\sum \alpha x_1 x_2 \dots x_k p_i(y_1, \dots, y_{n_i}) z_1 z_2 \dots z_l = 0,$$

where $\alpha \in \Omega$, and x, y, z with indices are arbitrary M-words (i.e., products of generators from M); moreover, in the notation of each polynomial $p_i(y_1, \dots, y_{n_i})$, generators from at least two different sets M_α occur. Then the algebra G, with system of generators M and system of defining relations $\Phi \cup \Psi$, is called the P-free product of the algebras $A_\alpha,\ \alpha \in S$, defined by the identity relations (P).

Definitions 1 and 2 are equivalent, which can be shown analogously to how this is done in group theory ((2), pp. 209–210).

Let us note the simplest properties of P-free products of algebras.

I. $A \nabla B = B \nabla A$.

II. If $G = \prod_{\alpha \in S}^\nabla A_\alpha$, if the set S is the union of pairwise disjoint subsets S_β: $S = \bigcup_\beta S_\beta$, and if $B_\beta = \prod_{\alpha \in S_\beta}^\nabla A_\alpha$, then

$$G = \prod_\beta^\nabla B_\beta.$$

III. If $G = \prod_\alpha^\nabla A_\alpha$ is the P-free product of the algebras A_α, and if in each algebra A_α a subalgebra B_α has been chosen, then the subalgebra B generated in G by all B_α will be their P-free product:

$$B = \prod_{\alpha}^{\nabla} B_{\alpha}.$$

IV. If $G = \prod_{\alpha}^{\nabla} A_{\alpha}$ and if some factors A_{α} themselves are decomposed into a P-free product:

$$A_{\alpha} = \prod_{\beta}^{\nabla} A_{\alpha\beta},$$

then

$$G = \prod_{\alpha,\beta}^{\nabla} A_{\alpha\beta}.$$

This latter decomposition of the algebra G we shall call a continuation of the original decomposition.

V. If $G = A \nabla B$ and $\bar{A}$ is the two-sided ideal generated in G by the subalgebra A, then $G/\bar{A} \cong B$.

Any algebra in which, for any of its elements $x_1, x_2, \ldots, x_{n_i}$, all equalities (P) hold is called a P-algebra. According to A. I. Mal' cev (3), an algebra with a system of generators M and a system of identity defining relations (P) is called a free P-algebra with system M of free generators.

VI. The P-free product of algebras A_{α} is a P-algebra if and only if each algebra A_{α} is a P-algebra.

VII. The P-free product of free P-algebras A_{α} with systems M_{α} of free generators is a free P-algebra with the system

$$M = \bigcup_{\alpha} M_{\alpha}$$

of free generators.

VIII. A free P-algebra with system $\{a_{\alpha}\}$ of free generators can be decomposed into the P-free product of free P-algebras A_{α} on one generator a_{α}.

IX. Every P-free product G of algebras A_{α} contains a two-sided ideal J which is a free P-algebra and such that G/J is the direct product of the same algebras A_{α}.

Let us give several examples of P-free products of algebras.

Example 1. The system (P)

$$xy + yx = 0,$$

$$(xy)z + (yz)x + (zx)y = 0$$

defines the left-free product of algebras.

Example 2. The reduced free products defined by the systems

$$xy - yx = 0, \tag{K}$$

$$xy + yx = 0, \tag{AK}$$

will be called, respectively, **commutative free** (K-free) and **anticommutative free** (AK-free) products of algebras.

Example 3. The system (P)

$$xy = 0$$

defines the direct product of algebras.

The nonassociative free product of algebras may be regarded as a P-free product, assuming that the system (P) is empty in this case.

2. In what follows we shall speak only about K-free and AK-free products of algebras. Therefore, in the expressions "P-free products," "P-algebras," the letter P is everywhere assigned one and the same meaning, K or AK.

We shall show how to find a basis B of an algebra G which is a P-free product of algebras A_α, if bases L_α of the algebras A_α, $\alpha \in S$, are given. Let $\mathfrak{G}$ be the free P-algebra with the set $L = \bigcup_{\alpha \in S} L_\alpha$ of free generators. According to (4), as a basis $\mathfrak{B}$ of the algebra $\mathfrak{G}$ one may take the ordered aggregate of all so-called P-regular words. The set of all those P-regular words which do not contain, in any pair of brackets, two elements of one and the same algebra A_α, forms precisely a basis B of the algebra G. An arbitrary element $x \in G$ can be expressed through the elements of the basis B in the following way. First we write x in regular form, regarding it as an element of the nonassociative product of the algebras A_α ((1), p. 245). Then, regarding each regular word occurring in this expression as an element of the above-mentioned free P-algebra $\mathfrak{G}$, we express them in P-regular form in the same way as A. I. Shirshov does this ((4), p. 82). As a result we obtain the desired expression of the element x through the basis B.

By modifying in the appropriate way the methods of proof of the theorem on subalgebras and Zhukov' s theorem for nonassociative free products of algebras, proposed by A. G. Kurosh ((5)), one can prove the following theorems.

Theorem 1. Let

$$G = \prod_{\alpha \in S} A_\alpha \, F$$

be the P-free product of arbitrary algebras A_α and a free P-algebra F, and let H be a subalgebra of the algebra G which is a P-algebra. Then

$$H = \prod_{\alpha \in S} B_\alpha \, V,$$

where $\prod$ is the same P-free product, $B_\alpha = H \cap A_\alpha$, and V is some free P-algebra.

A special case of this theorem, when all $A_\alpha = 0$, is the theorem of A. I. Shirshov that every subalgebra of a free P-algebra is itself free (4). (The condition that H must be a P-algebra is essential.)

Let

$$g_1, \ g_2, \dots, \ g_n \tag{2}$$

be some finite system of generators of the algebra G. According to A. G. Kurosh ((5), p. 257), **elementary transformations of the system** (2) are transformations of one of the following two types:
a) addition to any g_i of an arbitrary polynomial in all the remaining elements of the system (2);
b) multiplication of an element g_i by some distin-

a nonzero number of elements of the ground field Ω. An elementary transformation of the system (2) again leads to a system of generators of the algebra G.

Theorem 2. *Let G be a P-algebra with a finite number of generators (2), and*

$$G = A_1 \nabla A_2 \nabla \cdots \nabla A_k$$

a P-free product. Then by elementary transformations the system (2) can be brought to such a form that all generators (2) will lie in the factors $A_1, \dots, A_k$; moreover, if $g_1, g_2, \dots, g_l$ are all the generators from the system (2) lying, for example, in A_1, then they generate A_1.

Corollary 1. Every P-algebra with a finite number of generators can be decomposed into the corresponding P-free product of a finite number of algebras indecomposable into such a product.

Corollary 2. P-free decompositions of a P-algebra with n generators consist of no more than n free factors.

Corollary 3. The minimal number of generators of a P-algebra with a finite number of generators is equal to the sum of the corresponding numbers for all factors of any decomposition of it into a P-free product.

In conclusion let us consider the question of the isomorphism of decompositions into a P-free product of algebras ($P = K$ or $P = AK$).

Definition 3. Two P-free decompositions of an algebra shall be called **isomorphic** if every free factor of one of these decompositions that is not a free P-algebra is a free factor in the other decomposition as well, and conversely, i.e., if

$$G = \prod_{\alpha}^{\nabla} A_\alpha \nabla F = \prod_{\beta}^{\nabla} B_\beta \nabla F',$$

where F, F' are free P-algebras, and A_α, B_β are nonfree P-algebras, then $A_\alpha = B_\alpha$ under a corresponding numbering.

By property V of P-free products, here $F \cong F'$.

Theorem 3. *Any two P-free decompositions of a P-algebra G possess isomorphic refinements. In particular, if a P-algebra G admits a P-free decomposition with indecomposable factors, then all such decompositions are isomorphic to one another.*

The proof of this theorem is carried out on the basis of Theorem 2 and properties I–VIII of P-free products literally in the same way as for nonassociative free decompositions ([1], pp. 258–259).

I express my gratitude to A. I. Mal' cev and A. I. Shirshov for their attention to the present work and for valuable comments.

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Received
6 IV 1960

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Note: Figure translations are in progress. See original paper for figures.