Reports of the Academy of Sciences of the USSR

MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1960. Vol. 132, No. 5

MATHEMATICS

A. F. LEONT' EV

ON THE CONVEXITY OF THE DOMAIN OF REGULARITY OF A SOLUTION OF A DIFFERENTIAL EQUATION OF INFINITE ORDER

(Presented by Academician I. M. Vinogradov on February 8, 1960)

Let

$$Dy = y^{(s)} + p_1(z)y^{(s-1)} + \cdots + p_s(z)y, \tag{1}$$

where $p_1(z), \ldots, p_s(z)$ are entire functions, and let

$$D^0 y = y(z), \qquad D^k y = D\left(D^{k-1}y\right).$$

Introduce the equation

$$M(y) \equiv \sum_0^\infty c_n D^n y = F(z). \tag{2}$$

Suppose that $F(z)$ is an entire function and that the constants c_m are such that the characteristic function

$$L(z) = \sum_0^\infty c_m z^m$$

belongs to the class $[1/s, 0]$, i.e. for every $\varepsilon > 0$ it satisfies the condition

$$|L(z)| < e^{\varepsilon |z|^{1/s}}, \qquad |z| > r_0(\varepsilon).$$

It can be shown (see, for example, ([1]), p. 205) that the series standing on the left-hand side of (2) converges at every point at which the function $y(z)$ is regular, and its sum is regular at that point.

Theorem. *The domain of existence of any solution of equation (2) is convex.*

For $Dy = y'$ we obtain from this Polya' s theorem (^{2}).

The proof of the theorem is based on the following lemma, which is also of independent interest. Denote by $y(z, \lambda)$ the solution of the equation

$$Dy = \lambda^s y$$

(here λ is a parameter) satisfying the initial conditions

$$y(z_0, \lambda) = 1, \qquad y'(z_0, \lambda) = \lambda, \dots, y^{(s-1)}(z_0, \lambda) = \lambda^{s-1}.$$

Lemma. If

$$\gamma(z) = \sum_0^\infty \frac{\alpha_m}{(z - z_0)^{m+1}}, \qquad \overline{\lim_{m\to\infty}} \sqrt[m]{|\alpha_m|} = \rho < \infty,$$

then the function

$$F(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1>\rho} \gamma(z) y(z, \lambda)\, dz$$

is an entire function of order one of type ρ.

For the proof of the lemma, put

$$y(z, \lambda) = \sum_0^\infty A_m(z) \lambda^m. \tag{3}$$

Here $A_m(z)$ are entire functions. In terms of them one may expand any function analytic in a neighborhood of the point z_0. Namely, M. K. Fage proved[3] that if $f(z)$ is regular in the disk $|z - z_0| < r$, then in some neighborhood of the point z_0 the following representation holds:

$$f(z) = \sum_{m=0}^\infty \left\{ \frac{d^q}{dz^q} D^p f \right\}_{z=z_0} A_m(z), \qquad m = ps + q, \quad 0 \le q < s. \tag{4}$$

If one takes into account M. K. Fage' s estimate for $|A_m(z)|$,

$$|A_m(z)| < C \frac{|z - z_0|^m}{m!} \tag{5}$$

(C is a constant, in general different for each disk), and the estimate for $\left|(D^p f)^{(q)}\right|$,

$$\left|(D^p f)^{(q)}_{z=z_0}\right| < \frac{m!}{r^m}\,\frac{M(R,f)}{(1-r/R)^{NR+1}}, \qquad 0<r<R, \tag{6}$$

$$N = \max_{1\le j\le s}[1, M(R;p_j)], \qquad M(R,\varphi) = \max_{|z-z_0|=R}|\varphi(z)|$$

(in the case $s=2$ it was obtained in paper[4]; in the general case it is proved analogously), then we obtain that the series (4) converges, in any event, in the disk $|z-z_0|<\rho$, whose radius ρ is equal to the distance from z_0 to the nearest singular point of $f(z)$.

Put

$$\frac{1}{2\pi i}\int_{|z-z_0|=\rho_1}\gamma(z)A_m(z)\,dz = \frac{\beta_m}{m!} \tag{7}$$

and show that

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|\beta_m|} = \rho. \tag{8}$$

Suppose the contrary, i.e., that

$$\overline{\lim_{m\to\infty}}\sqrt[m]{|\beta_m|} = \mu < \rho \tag{9}$$

(this limit, by virtue of (5), cannot be greater than ρ). Consider the function

$$\Phi(\lambda) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1}\gamma(z)e^{\lambda(z-z_0)}\,dz.$$

This function is entire of order one and type ρ. On the basis of (4) we have

$$e^{\lambda(z-z_0)} = \sum_{m=0}^{\infty}\left[D^p e^{\lambda(z-z_0)}\right]^{(q)}_{z=z_0} A_m(z).$$

Consequently, taking (7) into account, we obtain

$$\Phi(\lambda) = \sum_{m=0}^{\infty}\frac{\beta_m}{m!}\,[D^p e^{\lambda(z-z_0)}]^{(q)}_{z=z_0}.$$

Let us estimate the right-hand side in modulus. To do this we use estimate (6), putting in it $f(z) = e^{\lambda(z-z_0)}$, $r = \mu + 2\varepsilon$, $R = \mu + 3\varepsilon$, and the condition (9), by virtue of which

$$|\beta_m| < (\mu+\varepsilon)^m, \qquad m > m_0(\varepsilon).$$

We shall have

$$|\Phi(\lambda)| < \frac{e^{|\lambda|(\mu+3\varepsilon)}}{\left(1 - \dfrac{\mu+2\varepsilon}{\mu+3\varepsilon}\right)^{NR+1}} \left\{ \sum_{m=0}^{m_0} \frac{|\beta_m|}{(\mu+2\varepsilon)^m} + \sum_{m_0+1}^{\infty} \left(\frac{\mu+\varepsilon}{\mu+2\varepsilon}\right)^m \right\} < e^{(\mu+4\varepsilon)|\lambda|},$$

$$|\lambda| > \lambda_0.$$

For small $\varepsilon > 0$ the number $\mu + 4\varepsilon < \rho$, and therefore $\Phi(\lambda)$ is an entire function of the first order of type less than ρ, which is impossible. Consequently, equality (8) is valid. By virtue of this equality, taking (3) into account, we obtain that the function

$$F(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} \gamma(z) y(z,\lambda)\, dz = \sum_{0}^{\infty} \frac{\beta_m}{m!} \lambda^m$$

is indeed an entire function of the first order of type ρ.

For what follows, along with Dy and $M(y)$, we introduce the operators

$$\widetilde{D}y = y^{(s)} + [p_1(z)y]^{(s-1)} + \cdots + p_s(z)y, \qquad \widetilde{M}y = \sum_{0}^{\infty} c_m \widetilde{D}^m y$$

and denote by $\tilde{y}(z,\lambda)$ the solution of the equation $\widetilde{D}y = \lambda^s y$, satisfying at the point z_0 the same conditions as $y(z,\lambda)$.

Proof of the theorem. Let $y(z)$ be a solution of equation (2). Suppose the contrary, i.e., that the domain G of existence of the function $y(z)$ is not convex. It will follow from this that through some boundary point ξ of the domain G one can draw such a circle C that some arc (α, β) of it has with the boundary of the domain G only one common point $\xi \neq \alpha, \beta$ and is turned with its convexity into the domain G (a proof of this fact is given in paper ([5]), p. 72; the corresponding figure is also given there). Join the points α and β in the domain G by two paths L_1 and L_2. Let, moreover, L_2 lie in the domain bounded by the curve L_1 and the arc (α, β) of the circle C. When the point z lies in the domain E bounded by the curves L_1 and L_2, by Cauchy' s formula we have

$$y(z) = \frac{1}{2\pi i}\int_{L_1}\frac{y(t)\,dt}{t-z} + \frac{1}{2\pi i}\int_{L_2}\frac{y(t)\,dt}{t-z} = y_1(z) + y_2(z).$$

The function $y_1(z)$ is regular everywhere outside L_1; in particular, it is regular at the point ξ. Using the equality $y_2(z) = y(z) - y_1(z)$, we shall continue the function $y_2(z)$ from the domain E through the curve L_2. We see that $y_2(z)$ is regular everywhere outside the circle C, while the point ξ is a singular point for it. Note also that $y_2(\infty) = 0$. Draw through the point ξ a circle C_1, tangent to C at the point ξ and enclosing C. The function $y_2(z)$ is regular outside C_1 and at all points of C_1 different from ξ. We shall show that the point ξ is singular for the function

$$M(y_2) = \sum_0^\infty c_m D^m y_2(z).$$

This function is regular outside C_1 and at all points of C_1 distinct from ξ. Let z_0 and ρ be, respectively, the center and the radius of the circle C_1. Outside C_1

$$y_2(z) = \sum_0^\infty \frac{a_m}{(z-z_0)^{m+1}}, \qquad \overline{\lim_{m\to\infty}}\sqrt[m]{|a_m|} = \rho.$$

We have

$$\frac{1}{2\pi i}\int_{|z-z_0|=\rho_1>\rho} D^m[y_2(z)]\tilde{y}(z,\lambda)\,dz = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} y_2(z)\widetilde{D}^m[\tilde{y}(z,\lambda)]\,dz,$$

whence

$$\omega(\lambda) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} M[y_2(z)]\tilde{y}(z,\lambda)\,dz = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} y_2(z)\widetilde{M}[\tilde{y}(z,\lambda)]\,dz.$$

Since

$$\widetilde{M}[\tilde{y}(z,\lambda)] = L(\lambda^s)\tilde{y}(z,\lambda),$$

it follows that

$$\omega(\lambda) = L(\lambda^s)\omega_1(\lambda), \qquad \omega_1(\lambda) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho_1} y_2(z)\tilde{y}(z,\lambda)\,dz.$$

By the lemma, $\omega_1(\lambda)$ is an entire function of order one of type ρ. By the hypothesis, $L(\lambda^s)$ is an entire function of class $[1, 0]$. Therefore $\omega(\lambda)$ is an entire function of order one of type ρ. From this, on the basis of the lemma, it follows that the function $M(y_2)$ has at least one singular point on the circle C_1. This point is the point ξ. It is now easy to arrive at a contradiction. Equation (2) gives

$$M(y_1) + M(y_2) = F(z).$$

The first term is regular at the point ξ, while the second term has a singularity at the point ξ; consequently, the point ξ is singular for $F(z)$, which is impossible. Hence the domain G is convex.

Moscow Power Engineering
Institute

Received
1 II 1960

CITED LITERATURE

[1] A. F. Leont' ev, *Izv. AN SSSR, Ser. Mat.*, **22**, 201 (1958).
[2] G. Polya, *Nachr. Gesellsch. Wissensch. Göttingen*, 187 (1927).
[3] M. K. Fage, DAN, **112**, No. 6, 1008 (1957).
[4] A. F. Leont' ev, *Izv. AN SSSR, Ser. Mat.*, **23**, 565 (1959).
[5] A. F. Leont' ev, *Tr. Matem. Inst. im. V. A. Steklova*, **39** (1951).

Note: Figure translations are in progress. See original paper for figures.