FLUCTUATIONS OF THE MAGNETIC FIELD AND CURRENT DENSITY IN A TURBULENT FLOW OF A WEAKLY CONDUCTING LIQUID

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Abstract

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FLUCTUATIONS OF THE MAGNETIC FIELD AND CURRENT DENSITY IN A TURBULENT FLOW OF A WEAKLY CONDUCTING LIQUID

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PHYSICS

1. A number of examples can be given of weakly conducting turbulent media (the lower ionosphere, the ocean) situated in an external magnetic field whose influence on the dynamics of the medium may be neglected. At the same time, motions of a conducting medium in a magnetic field will induce currents. As a result, under turbulent motions the magnetic field will fluctuate slightly, and the intensity of these pulsations is evidently determined by the magnitude of the conductivity and the intensity of the turbulence. The present note is devoted to calculating the spectrum of fluctuations of the magnetic field, as well as the spectrum and structure function of the density of the induced currents.
2. The equations for the magnetic field **H** in a conducting medium moving with velocity **v** have the form:

$$\partial H_i/\partial t + v_k \partial H_i/\partial x_k - H_k \partial v_i/\partial x_k = v_m \partial^2 H_i/\partial x_k^2; \qquad \partial v_k/\partial x_k = 0; \qquad \partial H_k/\partial x_k = 0, \tag{1}$$

where $v_m = c^2/4\pi\sigma$; c is the speed of light; σ is the conductivity of the medium. The coefficient v_m is the magnetic viscosity, which determines the dissipation and diffusion of the magnetic field in the medium. By weak conductivity we shall mean the condition $v_m \gg v$, where v is the kinematic viscosity of the medium.* We shall assume the turbulence to be locally homogeneous and locally isotropic.

Represent the magnetic field in the form $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$, where $\mathbf{H}_0$ is the principal, external, stationary magnetic field, and $\mathbf{H}'$ is the pulsation of the field, with $|\mathbf{H}'| \ll H_0 = |\mathbf{H}_0|$. Estimating the terms of equation (1) by order of magnitude, we obtain, in the first approximation with respect to $h = |\mathbf{h}| = |\mathbf{H}'|/H_0$, the equation

$$-\lambda_k \partial v_i/\partial x_k = v_m \partial^2 h_i/\partial x_k^2, \tag{2}$$

where $\vec{\lambda} = \mathbf{H}_0/H_0$. This problem is, in a certain sense, analogous to the problem of fluctuations of a passive scalar admixture in a turbulent medium with a large diffusion coefficient k, considered in work (^{1}), in which the corresponding fluctuation spectrum was found in the interval of wave numbers $(\varepsilon/k^3)^{1/4} \ll p \ll (\varepsilon/v^3)^{1/4}$, where ε is the dissipation of turbulent energy per unit mass. This interval lies between the characteristic scale of dissipation for the admixture and the scale of viscous dissipation of the kinetic energy of turbulent motions. From arguments analogous to those used in (^{1}), it follows that the time derivative may be neglected even in comparison with the second term on the left in (1), which is discarded. Let us carry out these arguments for our case.

One may think that equation (1) is equivalent to the equation of heat conduction in a solid body with thermal diffusivity coefficient v_m and with distributed sources described by the last two terms on the left in (1). If the sources are stationary, then for arbitrary— under arbitrary initial conditions, the adjustment time of a field perturbation of scale l to the distribution of sources is of the order of $\nu_m^{-1/2}$. In the inertial range of turbulence the characteristic time of variation of the sources is equal to $l(\varepsilon l)^{-1/3}$, which is much greater than $\nu_m^{-1/2}$. This at once gives an upper bound on the scale of the phenomena under consideration. It must be $l \ll (\nu_m^3/\varepsilon)^{1/4} = l_m$. For scales of the order of the inner scale of turbulence $l_0 = (\nu^3/\varepsilon)^{1/4}$ and smaller, the ratio of the corresponding times is of the order of ν_m/ν. Thus the sources are approximately stationary, i.e., the field rapidly adjusts to the change in the equivalent sources, which also determine the spatial distribution of the field throughout the whole range of scales smaller than l_m. Beginning with the scale l_m, the energy of the fluctuations of the magnetic field in the turbulent medium is dissipated into Joule heat. If $l_m < L$, where L is the outer scale of the turbulence, then the range of scales under consideration includes the similarity interval (inertial interval) for the velocity pulsations, in which the Kolmogorov –Obukhov "2/3 law" [2] holds, as well as the interval of viscous dissipation. In the case $L < l_m$, the limiting scale from above is L.

Let us determine the conditions under which the influence of the field on the character of the turbulence may be neglected. For this purpose, in the equation of motion one must compare, in order of magnitude, the gradient of the turbulent pressure pulsations [3] (corresponding to the distance l between observation points) $\nabla p \simeq \rho(\varepsilon l)^{2/3} l^{-1}$ with the volumetric electromagnetic force arising during turbulent motions of a conducting medium in a magnetic field, whose order of magnitude may be estimated with the aid of equation (2)

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* This inequality is true with a large margin for almost all media, with the exception of a very rarefied gas. Thus, in the ionosphere at an altitude of 100 km, $v \simeq 2 \cdot 10^5$ cm^2 sec^{-1}, $v_m \simeq 2 \cdot 10^{15}$ cm^2 sec^{-1}.

$$c^{-1}[\mathbf{jH}] = (4\pi)^{-1}[\operatorname{rot}\mathbf{H}\cdot\mathbf{H}] = (4\pi)^{-1}H_0^2[\operatorname{rot}\mathbf{h}\cdot\vec{\lambda}] \simeq (4\pi\nu_m)^{-1}H_0^2(\varepsilon l)^{1/3}.$$

The ratio of this term to the pressure gradient is, in order of magnitude,

$$(4\pi\nu_m)^{-1}H_0^2(\varepsilon l)^{1/3} : \rho(\varepsilon l)^{2/3}l^{-1} = V_A^2\nu_m^{-1}l(\varepsilon l)^{-1/3}; \qquad V_A^2 = H_0^2/4\pi\rho. \tag{3}$$

If this ratio is much less than unity, then the influence of the field on turbulent eddies of scale l may be neglected. Denote by l_H the limiting scale for which the ratio (3) is equal to unity. Then the condition for absence of influence of the field on the character of the turbulence will be the requirement that this scale be greater than l_m:

$$l_H/l_m = \varepsilon l_m/V_A^3 > 1 \quad \text{or} \quad l_m > V_A^3/\varepsilon. \tag{4}$$

3. Let us turn to the calculation of the spatial spectrum of the fluctuations of the magnetic-field strength vector. The fluctuation field $\mathbf{h}$ will be an axisymmetric random field because of the presence of the distinguished direction $\mathbf{H}_0$. From (2) one can obtain an equation for the structure function of the magnetic-field fluctuations $M_{ij} = \nu_m^2[h_i(A)-h_i(A')][h_j(A)-h_j(A')]$, analogous to the way in which the equation for the structure function of the pressure pulsations [3] was obtained:

$$\partial^4 M_{ij}(\mathbf{r})/\partial\xi_k^2\partial\xi_l^2 = -\lambda_k\lambda_l\partial^2 D_{ij}(\mathbf{r})/\partial\xi_k\partial\xi_l. \tag{5}$$

Here $\mathbf{r}$ is the vector joining the observation points A and A'; ξ_i is a component of $\mathbf{r}$; $r = |\mathbf{r}|$; D_{ij} is the structure function of the velocity field. Using the spectral expansions of the structure functions of the velocity and magnetic-field fluctuations [4]:

$$D_{ij}(\mathbf{r}) = 2\int_{-\infty}^{\infty}\!\!\iint(1-e^{i\mathbf{pr}})\Phi_{ij}(\mathbf{p})\,d\mathbf{p}; \qquad M_{ij}(\mathbf{r}) = 2\int_{-\infty}^{\infty}\!\!\iint(1-e^{i\mathbf{pr}})F_{ij}(\mathbf{p})\,d\mathbf{p}$$

and substituting them into (5), we readily obtain

$$F_{ij} = \frac{\lambda_k\lambda_l p_{kp}l}{p^4}\Phi_{ij}(\mathbf{p}) = \frac{\mu^2 E(p)}{4\pi p^4}\left(\delta_{ij} - \frac{p_{ip}j}{p^2}\right) = \frac{E_H(p,\mu)}{4\pi p^2}\left(\delta_{ij} - \frac{p_{ip}j}{p^2}\right), \tag{6}$$

where $\mu = \lambda_i p_i/p$; $E(p)$ is the single scalar function determining the spectral tensor of the velocity field (5), and $E_H(p,\mu)$ is the corresponding function for the magnetic-field spectrum. In the inertial range of wave numbers (6),

$$E(p) = (6\pi)^{-1} 11\Gamma(8/3)\sin(\pi/3) C_v^2 p^{-5/3} = 0.77 C_v^2 p^{-5/3},$$

where $C_v^2 = C\varepsilon^{2/3}$ is the structural constant in the "2/3 law" for the velocity; C is a dimensionless constant, of order unity. Comparison of $E_H(p, \mu)$ with $E(p)$ shows an angular anisotropy of the former, as well as a more rapid decrease with increasing wave number, namely $E_H \sim p^{-11/3}$. Let us note that the corresponding spectral density for fluctuations of a passive scalar admixture in the case of a large diffusion coefficient (1) decreases as $p^{-17/3}$. The slower decrease in our case is connected with the presence of a constant external field, which is absent in the case of a passive scalar admixture. Let us also note that such a rapid decrease of the spectrum does not allow one to determine, in this interval, an automodel structural function for the magnetic-field fluctuations. For this it is necessary (4) that the spectral density decrease no faster than p^{-3}.

4. However, for derivatives of the field **h**, in particular for the vector **j** of current density induced by random motions of the conducting medium in a magnetic field, the structural function can be determined, i.e., the fluctuation field **j** will already be a locally homogeneous random field.

Taking into account that $\mathbf{j} = (4\pi)^{-1} c \operatorname{rot} \mathbf{H}$, we obtain an equation for determining the structural function of the current-density fluctuations $I_{ij} = 16\pi^2 c^{-2} v_m^2 [j_i(A) - j_i(A')][j_j(A) - j_j(A')]$, which will have the form

$$\varepsilon_{ikl}\varepsilon_{jmn}\partial^2 I_{ln}/\partial\xi_k\partial\xi_m = \delta_{ij}\Delta I_{ll} - \partial^2 I_{ll}/\partial\xi_i\partial\xi_j - \Delta I_{ij} = \lambda_k\lambda_l\partial^2 D_{ij}/\partial\xi_k\partial\xi_l. \quad (7)$$

The procedure for obtaining it is again analogous to that used in (3) and in the preceding item. Hence the spectral tensor Ψ_{ij} of the current density is

$$\Psi_{ij} = \frac{\lambda_k\lambda_l p_{kp} l}{p^2}\Phi_{ij}(\mathbf{p}) = \frac{\mu^2 E(p)}{4\pi p^2}\left(\delta_{ij} - \frac{p_{ip} j}{p^2}\right). \quad (8)$$

Thus, the order of decrease of the spectrum for the current density is the same as for the velocity spectrum.

To determine the structural function I_{ij}, let us first calculate the right-hand side of equation (7). Using the fact that in the inertial range

$$D_{ij} = \frac{4}{3} r^{2/3}\delta_{ij} - \frac{1}{3} r^{-4/3}\xi_i\xi_j$$

(for the present the structural constant is taken equal to unity), by successively differentiating and summing we obtain:

$$\lambda_k\lambda_l\frac{\partial^2 D_{ij}}{\partial\xi_k\partial\xi_l} = r^{-4/3}\left\{\frac{4}{9r^2}\left(1 - \frac{10}{3}m^2\right)\xi_i\xi_j - \frac{2}{3}\lambda_i\lambda_j +\right.$$

$$+\frac{8}{9}\left(1-\frac{4}{3}m^2\right)\delta_{ij}+\frac{8m}{9r}(\xi_i\lambda_j+\xi_j\lambda_i)\Big\}, \tag{9}$$

where $m = \lambda_i\xi_i/r$. On the right we have an axisymmetric tensor of rank 2 of general form (5), symmetric in i and j. Consequently, the structural tensor must be sought in the same form:

$$I_{ij} = a\xi_i\xi_j + b\lambda_i\lambda_j + c\delta_{ij} + d(\xi_i\lambda_j + \xi_j\lambda_i), \tag{10}$$

where a, b, c, d are unknown functions of r and m, which must be determined using equation (7).

Let us first determine the trace I_{ii} of the structural tensor, which obeys the simpler equation:

$$\Delta I_{ii} = \lambda_k\lambda_l\partial^2 D_{ii}/\partial\xi_k\partial\xi_l. \tag{11}$$

Expanding the axisymmetric Laplace operator and performing the summation on the right, we obtain

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1-m^2}{r^2}\frac{\partial^2}{\partial m^2}-\frac{2m}{r^2}\frac{\partial}{\partial m}\right)I_{ii}=\frac{22}{9}r^{-4/3}\left(1-\frac{4}{3}m^2\right). \tag{12}$$

Starting from the form of the right-hand side of (7), it is natural to seek I_{ii} in the form $I_{ii} = r^{2/3}f(m)$. Substituting this expression into (12), we obtain an equation for determining the function $f(m)$:

$$(1-m^2)f''-2mf'+\frac{10}{9}f=\frac{22}{9}(1-\frac{4}{3}m^2).$$

A particular solution of this inhomogeneous equation is the function $f(m) = 1 + 2m^2/3$. The general solution of the homogeneous equation is a combination of two linearly independent Legendre functions; moreover, it can be shown that, for the given values of the coefficients, there are no solutions of our equation bounded on the interval $|m| \leqslant 1$, and therefore they must be discarded. As a result we obtain:

$$I_{ii} = r^{2/3}(1 + 2m^2/3). \tag{13}$$

To determine the tensor I_{ij}, we substitute into (7) the general form of I_{ij} given by formula (11), and equate the coefficients on the right and on the left at $\xi_i\xi_j$, $\lambda_i\lambda_j$, etc. We then obtain a system of differential equations for determining the functions a, b, c, d. The solution of these equations is carried out successively, exactly as was just done to determine the form of I_{ii}. Knowing it facilitates the solution process. The calculations give:

$$a = \frac{r^{-4/3}}{116}\left(\frac{11}{31} - \frac{35}{3}m^2\right); \quad b = \frac{3}{5}r^{2/3}; \quad c = \frac{r^{2/3}}{116}\left(\frac{2379}{155} + \frac{31}{3}m^2\right); \quad d = \frac{1}{4}r^{-1/3}m. \tag{14}$$

In the final formulas, all relations for determining the tensor I_{ij} should still be multiplied by the omitted structural constant C_v^2.

5. Some of the results obtained in this note have been applied to the ionosphere. Establishing the fact that the density of random currents, like velocity or a passive admixture, is a local quantity in the sense of the theory of locally homogeneous turbulence (i.e., that one can locally, independently of the external scale, determine the mean-square value of the difference at two points) made it possible to estimate the height up to which the influence of the Earth' s magnetic field on the character of turbulent motions in the ionosphere may be neglected. An estimate similar to (3), using data on the turbulence parameters in the lower ionosphere [7], shows that this influence is insignificant up to a height of ~ 150 km. The root-mean-square value of the difference (of the moduli) of the current density at a distance of 1 km at a height of 100 km is found to be of order 10^{-13} a $\cdot$ cm^{-2}.

Equation (2) makes it possible, knowing the correlation scale of the velocity vortices [7], to roughly estimate possible orders of magnitude of the mean square of magnetic-field pulsations at the Earth' s surface caused by turbulent motions in the lower ionosphere. Calculations show that $(H^2)^{1/2}$ may lie in the range $10^{-2} \div 1\ \gamma$ ($\gamma = 10^{-5}$ gs), which is smaller than the usually observed amplitudes of pulsations, although still accessible to measurement.

Consider mercury, for which $\nu = 1.15 \cdot 10^{-3}$ cm^2 sec^{-1}, $\nu_m = 7.5 \cdot 10^3$ cm^2 sec^{-1}. If we assume that $H_0 = 300$ gs, $\varepsilon = 10^{-4}$ cm^2 sec^{-3} (for comparison, we note that at a height of 100 km $\varepsilon \sim 10^3$, in the troposphere $\varepsilon \sim 10$, in the ocean $\varepsilon \sim 10^{-2}$ cm^2 sec^{-3}), then estimate (3) shows that the influence of the magnetic field is insignificant for vortices with sizes smaller than 10^5 cm. For the root-mean-square value of the difference in current density at a distance of 10 cm we obtain a value of 3 ma $\cdot$ cm^{-2}.

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Note: Figure translations are in progress. See original paper for figures.