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Abstract

Full Text

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THE PLANCHEREL MEASURE FOR THE QUOTIENT SPACE

$SL(n;R)/SO(n;R)$

(Presented by Academician I. G. Petrovskii, 22 III 1960)

1. Let G be a connected real semisimple Lie group with finite center, and let K be its maximal compact subgroup (we shall denote elements of the group G by g, and of the group K by k). Further, let T_g^ρ be an irreducible unitary representation of the group G, acting in a Hilbert space H, where ρ is the parameter determining this representation. The representation T_g^ρ is called a **representation of class I** if in H there exists a vector ξ_0 satisfying the condition $T_k\xi_0 = \xi_0$.* Let T_g^ρ be a representation of class I, and let $\xi_0 \in H$ be a vector of unit length satisfying the condition $T_g^\rho \xi_0 = \xi_0$ when $g \in K$. Denote by $f_\rho(g)$ the function

$$f_\rho(g) = (\xi_0, T_g^\rho \xi_0). \tag{1}$$

The function $f_\rho(g)$ is called a **zonal spherical function** ($^{1-3}$) belonging to the representation T_g^ρ. We note that

$$f_\rho(k_1 g k_2) = f_\rho(g).$$

It is known that every function having this property can be expanded into an integral over the functions (1)

$$f(g) = \int a(\rho) f_\rho(g)\, d\mu(\rho),$$

where

$$\int |f(g)|^2\, dg = \int |a(\rho)|^2\, d\mu(\rho).$$

The aim of the present work is to compute the Plancherel measure $d\mu(\rho)$ for the case when G is the group of all real unimodular transformations of n-dimensional space.

2. We shall describe the integral representation, needed in what follows, of these zonal spherical functions (^{1,5}).** Denote by X the group of lower triangular matrices with ones on the diagonal

$$x = \begin{Vmatrix} 1 & 0 & 0 & \cdots & 0 \\ x_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ x_{n1} & x_{n2} & x_{n3} & \cdots & 1 \end{Vmatrix} \qquad (x \in X).$$

Let $x_1, x_2, \dots, x_n$ be its column vectors, and let $D_p(x)$ be the Gram determinant of the first p columns of the matrix x.

Spherical functions are functions on the group of diagonal matrices with positive entries:

$$\varepsilon = \begin{Vmatrix} \varepsilon_1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \varepsilon_n \end{Vmatrix}, \qquad \varepsilon_1 > 0, \dots, \varepsilon_n > 0; \qquad \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = 1.$$

Consider the matrix $\varepsilon x \varepsilon^{-1}$, equal to

$$\xi = \begin{Vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \xi_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ \xi_{n1} & \xi_{n2} & \xi_{n3} & \cdots & 1 \end{Vmatrix}, \qquad \text{where} \quad \xi_{ij} = x_{ij} \frac{\varepsilon_j}{\varepsilon_i}.$$

Let $\vec{\xi}_1, \vec{\xi}_2, \dots, \vec{\xi}_n$ be the columns of the matrix $\varepsilon x \varepsilon^{-1}$, and let $\Delta_p(\xi)$ be the determinant of the Gram matrix of the first p columns of the matrix $\varepsilon x \varepsilon^{-1}$. Thus,

$$D_p(x) = \begin{vmatrix} (x_1 x_1) & \cdots & (x_1 x_p) \\ \vdots & & \\ (x_{px} 1) & \cdots & (x_{px} p) \end{vmatrix},$$

$$\Delta_p(\xi) = \begin{vmatrix} (\vec{\xi}_1 \vec{\xi}_1) & \cdots & (\vec{\xi}_1 \vec{\xi}_p) \\ \vdots & & \\ (\vec{\xi}_p \vec{\xi}_1) & \cdots & (\vec{\xi}_p \vec{\xi}_p) \end{vmatrix} \qquad (p = 1, \dots, n).$$

It is easy to see that

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* It is known that, up to a scalar factor, the vector ξ_0 is unique.

** These functions have been computed explicitly on the manifold G/K when G is a complex semisimple group (^{1}) and on Grassmann manifolds (^{4}).

$$D_1 \geq 1, \dots, D_{n-1} \geq 1, \qquad D_n = 1.$$

Denote by dx the product of all differentials dx_{ij}:

$$dx = \prod_{i>j} dx_{ij}.$$

Theorem 1. *The zonal spherical functions of positive-definite type on the symmetric space $SL(n;\mathbb{R})/SO(n;\mathbb{R})$ are given by the integrals*

$$\Phi_\lambda(\varepsilon) = \varepsilon_1^{i\lambda_1-\frac{n-1}{2}} \varepsilon_2^{i\lambda_2-\frac{n-2}{2}} \cdots \varepsilon_{n-1}^{i\lambda_{n-1}+\frac{n-2}{2}} \varepsilon_n^{i\lambda_n+\frac{n-1}{2}} \times$$

$$\times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\Delta_1^{i\frac{\lambda_1-\lambda_2}{2}-\frac{1}{2}} \cdots \Delta_{n-1}^{i\frac{\lambda_{n-1}-\lambda_n}{2}-\frac{1}{2}}}{D_1^{i\frac{\lambda_1-\lambda_2}{2}+\frac{1}{2}} \cdots D_{n-1}^{i\frac{\lambda_{n-1}-\lambda_n}{2}+\frac{1}{2}}} [dx], \tag{2}$$

where

$$[dx] = \frac{1}{c}\, dx, \qquad c = \prod_{p=1}^{n-1} \left[B\left(\frac{p}{2}; \frac{1}{2}\right)\right]^{n-p}, \tag{3}$$

$\lambda_1, \dots, \lambda_n$ are real numbers.

The constant c is chosen so that

$$\int_X \frac{[dx]}{D_1 D_2 \cdots D_{n-1}} = 1.$$

Thus,

$$c = \int_X \frac{dx}{D_1 D_2 \cdots D_{n-1}}.$$

Zonal spherical functions in integral form are known from the works of I. M. Gelfand and M. A. Naimark (^{1,2}). The proof of Theorem 1 is based on one result of Harish-Chandra (see (^{5}), No. 2, Corollary 1 to Lemma 44 (p. 288)).

3. In the work (^{5}) it is proved that the Plancherel measure on the symmetric space G/K is closely connected with the asymptotic behavior of zonal spherical functions on this space. Namely, the zonal spherical function $f_\rho(h)$, as $h \to \infty$ on the Cartan subgroup, has the asymptotic form

$$f_\rho(h) \simeq c(\rho)e^{i(\rho,t)}$$

and the Plancherel measure $d\mu$ for the space G/K is given by the formula

$$d\mu = \frac{1}{|c(\lambda)|^2}\, d\lambda,$$

where $d\lambda$ denotes Euclidean measure on the l-dimensional real Euclidean space which parametrizes the family of zonal spherical functions of positive-definite type on the symmetric space G/K ($l = \operatorname{rank} G/K$). Everything reduces to finding the function $c(\lambda)$.

In the case under consideration, $\varepsilon_j/\varepsilon_i \to 0$ for $i > j$, and, therefore, $\xi_{ij} \to 0$. Hence it follows:

Theorem 2.

$$c(\lambda) = \frac{1}{c} C_n(\lambda),$$

where

$$C_n(\lambda) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} D_1^{-i\frac{\lambda_1-\lambda_2}{2}-\frac{1}{2}} \cdots D_{n-1}^{-i\frac{\lambda_{n-1}-\lambda_n}{2}-\frac{1}{2}}\, dx, \tag{4}$$

$$\lambda = (\lambda_1, \dots, \lambda_n).$$

The coefficient c is determined by formula (3).

The evaluation of the integral (4) will be carried out inductively. Indeed, the following theorem holds:

Theorem 3. *The coefficients $C_n(\lambda_1, \dots, \lambda_n)$ and $C_{n-1}(\lambda_1, \dots, \lambda_{n-1})$, corresponding to the groups $SL(n;\mathbb{R})$ and $SL(n-1;\mathbb{R})$, respectively, are connected by the relation*

$$C_n(\lambda) = B\left(i\frac{\lambda_1-\lambda_n}{2}; \frac{1}{2}\right) B\left(i\frac{\lambda_2-\lambda_n}{2}; \frac{1}{2}\right) \cdots$$

$$\cdots B\left(i\frac{\lambda_{n-1}-\lambda_n}{2}; \frac{1}{2}\right) C_{n-1}(\lambda_1, \dots, \lambda_{n-1}).$$

Let us note that, if r takes real values,

$$\left|B\left(ir; \frac{1}{2}\right)\right|^2 = \frac{|\Gamma(1/2)|^2 |\Gamma(ir)|^2}{|\Gamma(ir+1/2)|^2} = \pi \frac{1}{r\ \operatorname{th}(r\pi)}.$$

From Theorem 3 the final result immediately follows:

Theorem 4. *The Plancherel measure in the case of the symmetric space* $SL(n;\mathbb{R})/SO(n;\mathbb{R})$ *is given by the formula*

$$d\mu = \frac{c^2}{\pi^{\frac{n(n-1)}{2}}} \prod_{1 \le p \le q \le n} \frac{\lambda_p - \lambda_q}{2} \operatorname{th} \frac{\lambda_p - \lambda_q}{2} \pi \, d\lambda_1 \cdots d\lambda_{n-1}$$

$$(\lambda_1 + \cdots + \lambda_n = 0).$$

The coefficient c *is determined by formula (3).*

The asymptotics of zonal spherical functions on $SL(n;R)/SO(n;R)$, for $n = 3$, was previously obtained by F. I. Karpelevich by another method.

In conclusion I take this opportunity to express my sincere gratitude to F. I. Karpelevich for his constant assistance, valuable guidance, and fruitful discussions.

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Note: Figure translations are in progress. See original paper for figures.

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