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Abstract

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MATHEMATICS

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ON TRACE FORMULAS FOR A DIFFERENTIAL SINGULAR STURM-LIOUVILLE OPERATOR

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I. M. Gel' fand, B. M. Levitan, and L. A. Dikii obtained identities for the eigenvalues of a regular Sturm-Liouville operator (for the bibliography see (^{1})). These may be interpreted as expressions for the regularized spectral traces of integral powers of the operator directly in terms of the operator. Such a formulation makes it possible to pose a similar problem also for an operator with continuous spectrum.

In the present work relations are obtained which express certain characteristics of the spectrum of the operator

$$Ly \equiv -y'' + q(x)y, \qquad 0 \le x < \infty, \qquad y(0) = 0$$

in terms of $q(x)$. These relations are analogous to identities for eigenvalues. The method of derivation is based on studying the properties of the denominator of the resolvent. It also proves convenient in the case of a regular operator.

1. We shall assume throughout that*

$$\int_0^\infty x\,|q(x)|\,dx < \infty.$$

The spectrum of the operator L consists of the half-axis $[0, \infty]$—the continuous spectrum—and a finite number of negative eigenvalues $\lambda_l = -\varkappa_l^2$ ($\varkappa_l > 0$; $l = 1, 2, \dots, m$). In connection with the operator L one often considers the function

$$M(s) = 1 + \int_0^\infty e^{isx} q(x) \varphi(x, s)\, dx = A(s) e^{i\eta(s)}$$

$$(s = \sigma + i\tau,\ 0 \leq \tau < \infty,\ -\infty < \sigma < \infty).$$

The function $\eta(\sigma)$ is the argument of $M(\sigma)$, the so-called limiting phase. Let R_λ be the resolvent of L. A superscript zero will denote the operator corresponding to $q(x) \equiv 0$.

Theorem 1. The operator $R_\lambda - R_\lambda^0$ has a trace** for $\arg\lambda \neq 0$ and $\lambda \neq \lambda_l$ $(l = 1, 2, \dots, m)$,

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = -\frac{d}{d\lambda}\ln M(\sqrt{\lambda}); \qquad 0 \leq \arg\sqrt{\lambda} \leq \pi.$$

Corollary.

$$M(\sqrt{\lambda}) = \det(E + qR_\lambda^0).$$

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* For the notation used by us and for properties of the operator L, see (²).
** For the notion of the trace of abstract operators, see (³).

For $q(x) \in L[0, \infty]$

$$\ln M(\sqrt{\lambda}) = \frac{1}{\pi}\int_0^\infty \frac{\eta(\sqrt{z})}{z - \lambda}\,dz + \sum_{l=1}^{m} \ln\frac{\lambda - \lambda_l}{\lambda} \qquad (\mathrm{Im}\sqrt{\lambda} > 0);$$

hence

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = -\int_{-\infty}^{\infty} \xi(t)\,d\frac{1}{t - \lambda}, \tag{α}$$

where

$$\xi(t) = \begin{cases} \dfrac{1}{\pi}\eta(\sqrt{t}), & t > 0, \\ -\displaystyle\int_{-\infty}^{t} \sum_l \delta(z - \lambda_l)\,dz, & t < 0. \end{cases}$$

In connection with another problem, formulas of type (α) were considered in the work of I. M. Lifshits (⁴). They were studied in detail for abstract operators by M. G. Krein (⁵). In our example, what is of interest is the connection of the function $\xi(t)$ with the limiting phase $\eta(k)$.

2. The derivation of relations similar to identities for eigenvalues uses the following lemmas:

Lemma 1. If $q(x) \in L[0, \infty]$, then for $0 < \operatorname{Re} z < 1/2$

$$\frac{\pi}{2z} \sum_{l=1}^{m} \varkappa_l^{2z} = \sin \pi z \cdot L(z) - \cos \pi z \cdot H(z),$$

where

$$H(z) = \int_0^\infty k^{2z-1} \eta(k)\, dk, \qquad L(z) = \int_0^\infty k^{2z-1} \ln A(k)\, dk.$$

Lemma 1 is a consequence of the analytic properties of the function $M(s)$ in the upper half-plane. It can be obtained by contour integration of the function

$$\frac{d}{ds} M(s)\, \frac{1}{M(s)}\, s^{2z}.$$

Lemma 2. Suppose that, for $x \geq 0$, there exists a continuous derivative $q^{(n)}(x)$ $(n \geq 1)$, and that $q^{(l)}(x)$ $(l = 0, \ldots, n)$ have finite limits (necessarily equal to zero) as $x \to \infty$. Then, uniformly in the upper half-plane, the asymptotic formulas hold

$$M(s)\big|_{|s| \to \infty} = 1 - \sum_{l=0}^{n} \frac{(-1)^{l+1}}{(2is)^{l+1}}\, V_l + o\left(\frac{1}{|s|^{n+1}}\right),$$

$$\ln M(s)\big|_{|s| \to \infty} = - \sum_{p=1}^{n+1} \frac{(-1)^p}{(2is)^p}\, Q_p + o\left(\frac{1}{|s|^{n+1}}\right),$$

whence, on the real axis,

$$\ln A(k)\big|_{|k| \to \infty} = - \sum_{\mu=1}^{\left[\frac{n+1}{2}\right]} \frac{(-1)^\mu}{(2k)^{2\mu}}\, Q_{2\mu} + o\left(\frac{1}{|k|^{n+1}}\right),$$

$$\eta(k)\big|_{|k| \to \infty} = \sum_{\mu=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^\mu}{(2k)^{2\mu+1}}\, Q_{2\mu+1} + o\left(\frac{1}{|k|^{n+1}}\right).$$

In these formulas $V_l = \lim_{\alpha \to \infty} V_l(\alpha)$, $V_l(\alpha)$ $(0 \leq \alpha)$ are subject to the recurrence relations

$$V_0(\alpha) = - \int_0^\alpha q(z)\, dz,$$

$$V_l(\alpha) = q^{(l-1)}(0) + \sum_{m=0}^{l-1} C_{l-1}^{m} \int_0^{\alpha} dz\, V_m(z) q^{(l-m-1)}(z) \qquad (l = 1, \dots, n+1),$$

$$Q_p = V_{p-1} + \sum_{j=1}^{p-1} \frac{j}{p} V_{p-j-1} Q_j.$$

This lemma can be proved by representing the dependence of $\varphi(x, s)$ on s in the expression for $M(s)$ by means of the transformation operator and using the integral equation for the transformation operator.

The asymptotic formulas for $\ln A(k)$ and $\eta(k)$ make it possible to study the analytic continuation of $H(z)$ and $L(z)$ from the strip $0 < \operatorname{Re} z < 1/2$ to the right by means of the usual techniques (see, for example, (6)). It then turns out that $H(z)$ has simple poles at the points $1/2$, $3/2$, ..., while the function $L(z)$ has simple poles at the points $1, 2, \dots$. The residues at the poles are expressed directly in terms of the Q_μ. Hence, as a result of the analytic continuation of the identity of Lemma 1, it follows that

Theorem 2. *Under the assumptions of Lemma 2 the formulas*

$$(-1)^{\mu} \sum_{l=1}^{m} \varkappa_l^{2\mu} + \frac{2\mu}{\pi} \int_0^{\infty} k^{2\mu-1} \left[\eta(k) - \sum_{l=0}^{\mu-1} \frac{(-1)^{l+1}}{(2k)^{2l+1}} Q_{2l+1} \right] dk = (-1)^{\mu} \frac{\mu}{2^{2\mu}} Q_{2\mu} \qquad \left(\mu = 1, 2, \dots \leq \frac{n}{2} \right);$$

$$(-1)^{\mu} \sum_{l=1}^{m} \varkappa_l^{2\mu+1} - \frac{2\mu+1}{\pi} \int_0^{\infty} k^{2\mu} \left[\ln A(k) - \sum_{l=1}^{\mu} \frac{(-1)^{l+1}}{(2k)^{2l}} Q_{2l} \right] dk = (-1)^{\mu} \frac{2\mu+1}{2^{2\mu+2}} Q_{2\mu+1} \qquad \left(\mu = 0, \dots \leq \frac{n-1}{2} \right)$$

The first series of formulas is analogous to the identities for eigenvalues; the second expresses the same relations, but in terms of the function $A(k)$.

With the aid of Lemma 2 we find

$$Q_1 = -\int_0^{\infty} q(z)\, dz, \qquad Q_2 = q(0),$$

$$Q_3 = q'(0) + \int_0^{\infty} q^2(z)\, dz, \qquad Q_4 = q''(0) - 2q^2(0).$$

The formulas of the first series for $\mu = 1, 2$ give

$$-\sum_{l=1}^{m}\varkappa_l^2+\frac{2}{\pi}\int_0^\infty t\left[\eta(t)-\frac{1}{2t}\int_0^\infty q(z)\,dz\right]dt=-\frac{1}{4}q(0)\qquad(\mu=1).$$

This relation had already been obtained in a paper by one of the authors (^{7}).

$$\sum_{l=1}^{m}\varkappa_l^4+\frac{4}{\pi}\int_0^\infty t^3\left[\eta(t)-\frac{1}{2t}\int_0^\infty q(z)\,dz-\left(\frac{1}{2t}\right)^3\left(q'(0)+\int_0^\infty q^2(z)\,dz\right)\right]dt=$$

$$=\frac{1}{8}(q''(0)-2q^2(0))\qquad(\mu=2).$$

A comparison of our results with the results of L. A. Dikii for the case of a finite interval reveals a complete analogy*.

3. Let us make some remarks concerning the case of a finite interval

$$ly\equiv-y''+p(x)y,\qquad 0\le x\le\pi;\qquad y(0)=y(\pi)=0;\qquad r_\lambda=(l-\lambda)^{-1}.$$

The denominator of the resolvent can here be connected with the entire function $\omega(\lambda)\equiv\omega(\pi;\lambda)$, where

$$-\omega''(x,\lambda)+p(x)\omega(x,\lambda)=\lambda\omega(x,\lambda),\qquad\omega(0,\lambda)=0,\qquad\omega'(0,\lambda)=1.$$

Outside the points of the spectrum,

$$\operatorname{Sp}r_\lambda=-\frac{d}{d\lambda}\ln\omega(\lambda).$$

There is the representation

$$\omega(\lambda)=\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\int_0^\pi\frac{\sin\sqrt{\lambda}(\pi-t)}{\sqrt{\lambda}}\,p(t)\omega(t,\lambda)\,dt.$$

It is convenient to use it in studying the asymptotics of $\omega(\lambda)$ and the asymptotics of the eigenvalues λ_l, determined by the zeros of $\omega(\lambda)$.

It is not difficult to prove

$$\sum_{l=j+1}^{\infty}\lambda_l^s=\left[\frac{1}{\pi}p\int_{-\infty}^{0}\frac{\frac{d}{d\lambda}\omega(\lambda)}{\omega(\lambda)}\lambda^s\,d\lambda-i\sum_{l=1}^{j}\lambda_l^s\right]e^{-i\pi s}\sin\pi s;$$

λ_l, $l = 1, \ldots, j$, are the negative eigenvalues; λ_l, $l = j+1, \ldots$, are the positive eigenvalues; $-1 < \operatorname{Re} s < -\frac{1}{2}$.

Analytic continuation of this formula leads, as above, to identities for the eigenvalues.

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* See formulas (6.3) and (6.4) of work (1). In making the comparison one must take into account that in (1) the odd-order derivatives of $q(x)$ are taken to be equal to zero at the endpoints of the interval.

Note: Figure translations are in progress. See original paper for figures.

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