Hydromechanics

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Abstract

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TOWARD THE CONSTRUCTION OF THE HYDRODYNAMICS OF A VISCOUS FLUID

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The relations of the hydrodynamics of a viscous fluid ([1]) are based on Newton' s law, which asserts that in the case of pure shear the rate of shear is directly proportional to the corresponding tangential stress

$$\gamma = \frac{1}{\mu}\tau, \tag{1}$$

where μ is the coefficient of viscosity.

Below we consider questions of constructing a general relation between the tensors of rates of deformation (ε_{ij}) and stresses (σ_{ij}) for an isotropic, incompressible viscous fluid for which Newton' s law (1) is valid.

Suppose that the medium under consideration is: 1) homogeneous, 2) incompressible, 3) isotropic. Suppose further that: 4) the medium under consideration behaves in a completely analogous manner when the sign of the stresses is changed to the opposite one, i.e., in this case the components of the rate of deformation likewise merely change sign. Suppose that: 5) there exists a quite definite relation between the tensors (ε_{ij}) and (σ_{ij}), characterized by the single constant μ. Finally: 6) we shall regard the medium as viscous, assuming the validity of Newton' s law (1).

Assumptions 1)–6) define an incompressible elastic body for which, under pure shear, Hooke' s law is valid, if (ε_{ij}) is understood as the strain tensor and μ as the shear modulus.

These considerations determine to some extent the following constructions: we shall assume that there exists some function $U(\sigma_{ij})$, called the potential of the rates of deformation, such that

$$\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}} \quad (i = j); \qquad 2\varepsilon_{ij} = \frac{\partial U}{\partial \sigma_{ij}} \quad (i \neq j). \tag{2}$$

Obviously, in the present case the assumption of the existence of the potential $U(\sigma_{ij})$ by no means signifies an assumption of the reversibility of work, analogous to what occurs, for example, in the theory of the plastic potential.

Definition (2) presupposes differentiability of the function U; however, a priori there are no grounds whatever to consider that this is in fact so. It is therefore expedient to introduce the concept of a generalized potential, following the ideas of Prager ([2]) and Koiter ([3]). Suppose that the surface of the function U consists of pieces of differentiable surfaces U_k $(k = 1, 2, \dots, n)$. If a state corresponds to the intersection of the surfaces U_k $(k = 1, 2, \dots, m;\ m \leq n)$, then

$$\varepsilon_{ij} = \lambda_k \frac{\partial U_k}{\partial \sigma_{ij}} \quad (i = j), \qquad \lambda_1 + \lambda_2 + \dots + \lambda_m = 1;$$

$$2\varepsilon_{ij} = \lambda_k \frac{\partial U_k}{\partial \sigma_{ij}} \quad (i \neq j), \qquad \lambda_k \geq 0, \tag{3}$$

where in (3) summation over the index k has been performed.

It is obvious that assumptions 1)–4) lead to the fact that

$$U = U(\Sigma_2, |\Sigma_3|), \tag{4}$$

where Σ_2, Σ_3 are, respectively, the second and third invariants of the stress-deviator tensor.

It is known that

$$\Sigma_2 = (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2);$$

$$\Sigma_3 = s_{11}s_{22}s_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - s_{11}\sigma_{23}^2 - s_{22}\sigma_{31}^2 - s_{33}\sigma_{12}^2,$$

where $s_{ii} = \sigma_{ii} - \sigma,\ \sigma = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$.

If, starting from (4), in accordance with (2) (or in accordance with (3), depending on the smoothness of the function U), one determines the relation ε_{ij}–σ_{ij}, then all the restrictions imposed by assumptions 1)–4) turn out to be satisfied.

Let us consider the restrictions which Newton' s law (1) imposes on the form of the function U. First of all we shall show that the relations used in the literature for the hydrodynamics of a viscous fluid ([1]), bearing the name of the generalized Newton law, are obtained under the particular assumption

$$U = U(\Sigma_2). \tag{5}$$

Indeed, according to (2), from (5) we obtain

$$\varepsilon_{11} = 2\frac{\partial U}{\partial \Sigma_2}(2\sigma_{11} - \sigma_{22} - \sigma_{33}), \dots,$$

$$\gamma_{12} = 2\varepsilon_{12} = 12\frac{\partial U}{\partial \Sigma_2}\sigma_{12}, \dots \tag{6}$$

The missing expressions are obtained by cyclic permutation of the indices. In pure shear

$$\sigma_{12} = \tau, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{31} = 0,$$

$$\gamma_{12} = 2\varepsilon_{12} = \gamma, \quad \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0. \tag{7}$$

Then from (6), (7), and (1) we obtain

$$12\mu\frac{\partial U}{\partial \Sigma_2} = 1, \qquad U = \frac{1}{12\mu}\Sigma_2. \tag{8}$$

Using (8), we transform expressions (6) to the form

$$\sigma_{11} = \sigma + 2\mu\varepsilon_{11}, \quad \sigma_{12} = \mu\varepsilon_{12}, \quad \sigma_{22} = \sigma + 2\mu\varepsilon_{22},$$

$$\sigma_{23} = \mu\varepsilon_{23}, \quad \sigma_{33} = \sigma + 2\mu\varepsilon_{33}, \quad \sigma_{31} = \mu\varepsilon_{31}. \tag{9}$$

Thus, the assumptions used in deriving the generalized Newton law are essentially equivalent to the hypothesis that the potential $U(\sigma_{ij})$ is independent of the third invariant Σ_3. However, the nature of this hypothesis cannot be justified within the framework of the assumptions 1)–6) that have been made. Strictly speaking, it does not follow a priori from anywhere that the function U does not depend on the third invariant Σ_3.

Let us consider the general case. According to (4) and (2) we obtain

$$\varepsilon_{11} = 2\frac{\partial U}{\partial \Sigma_2}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) + \frac{\partial U}{\partial |\Sigma_3|}\operatorname{sign}\Sigma_3\left(s_{22}s_{33} - \sigma_{23}^2 + \frac{1}{18}\Sigma_2\right),$$

$$\gamma_{12} = 2\varepsilon_{12} = 12\frac{\partial U}{\partial \Sigma_2}\sigma_{12} + 2\frac{\partial U}{\partial |\Sigma_3|}\operatorname{sign}\Sigma_3(\sigma_{23}\sigma_{31} - s_{33}\sigma_{12}). \tag{10}$$

Fig. 1

Figure 1: Fig. 1

In the case of pure shear, from (7) and (10) we obtain

$$12\mu\frac{\partial U}{\partial \Sigma_2} = 1, \qquad \frac{\partial U}{\partial |\Sigma_3|} = 0. \tag{11}$$

Under pure shear $\Sigma_2 = 6\tau^2,\ \Sigma_3 = 0$; therefore it is sufficient to represent the function U as the sum of two terms

$$U = \frac{\Sigma_2}{12\mu} + \Phi(\Sigma_2, |\Sigma_3|), \tag{12}$$

where it must hold that

$$\frac{\partial \Phi}{\partial \Sigma_2} = \frac{\partial \Phi}{\partial |\Sigma_3|} = 0 \quad \text{for} \quad \Sigma_3 = 0.$$

The relations of the constitutive law obtained from (12) by means of (2) will satisfy all conditions 1)–6).

If one restricts oneself to the class of functions U that are unbent relative to the origin of coordinates, then the possible level surfaces of the potential U, for a given magnitude of the shear stress, lie in the deviatoric plane of the principal stresses between the hexagons shown in Fig. 1. One can construct relations of the hydrodynamics of a viscous fluid corresponding, for example, to the vertices of the hexagons in Fig. 1, in the same way as is done in the theory of plasticity ([4]).

Fig. 1

Thus, within the assumptions of a homogeneous, isotropic, incompressible continuum for which, in the case of pure shear, Newton' s law holds, it is possible, generally speaking, to construct infinitely many constitutive relations ε_{ij}—σ_{ij}.

Figuratively speaking, Newton' s law (1) is a certain characteristic through which an infinite set of integral surfaces can be drawn, corresponding to the constitutive law ε_{ij}—σ_{ij} in the general case. This class of relations should be called possible; the problem of determining the "true" relation ε_{ij}—σ_{ij} should be posed as a reasonably formulated variational problem.

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References

[1] G. Lamb, *Hydrodynamics*, 1947.
[2] W. Prager, *J. Appl. Mech.*, **20**, 317 (1953).
[3] W. T. Koiter, *Quart. Appl. Math.*, **11**, 350 (1953).
[4] D. D. Ivlev, *Prikl. matem. i mekh.*, **22**, no. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.