ON THE THEORY OF STEADY WAVES OF NONSMALL AMPLITUDE

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Abstract

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ON THE THEORY OF STEADY WAVES OF NONSMALL AMPLITUDE

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The first proof of the existence of a solution to the problem of waves of steady form was given by A. I. Nekrasov. The solution was represented in the form of a series in a small parameter, whose convergence was shown with the aid of majorant functions. Subsequently the problem was considered by many authors; in this connection the method of proof was considerably simplified, for example by using the theory of bifurcation points of nonlinear operator equations ([2]), or by reduction to Lyapunov—Schmidt integral equations ([3]). However, the results obtained in this way naturally applied only to waves of small amplitude, when the motion of the fluid is sufficiently close to an undisturbed flow.

The methods of the theory of positive operators make it possible to show the existence of a solution of A. I. Nekrasov' s problem without the assumption of its closeness to the trivial one; these same methods turn out to be useful in the study of periodic flows over a wavy bottom.

We shall consider in greater detail the problem of waves in an infinitely deep fluid and shall confine ourselves to stating the results for other problems, since they admit essentially analogous treatment.

1. Thus, suppose that we have such a steady irrotational motion of a fluid in which, at infinite depth, the fluid moves with constant velocity c, directed horizontally, while the profile of the free boundary is a fixed periodic curve. The problem of symmetric waves then leads to finding a periodic function $\Phi(\theta) \neq 0$, with period 2π, satisfying the equation

$$\Phi(\theta) = \frac{g\lambda}{2\pi c^2} \int_0^\pi K(\varepsilon\theta) e^{3C\Phi} \sin\Phi \, d\varepsilon.$$

Here λ is the wavelength; g is the acceleration of gravity; $K(\varepsilon\theta)$ is given by the relation

$$K(\varepsilon\theta) = 2\sum_{n=1}^{\infty} \frac{\sin n\varepsilon \sin n\theta}{n};$$

$C\Phi$ is the function conjugate to Φ and satisfying the condition

$$\int_0^{2\pi} C\Phi(\varepsilon)\, d\varepsilon = 0.$$

The unknown function $\Phi(\theta)$ is the angle made by the tangent to the wave profile with the horizontal. Let $\theta = 0$ correspond to the trough, and $\theta = \pi$ to the crest of the wave. We shall assume that each wave has one crest and one trough, i.e.

a) $\Phi(\theta) \geq 0$ for $0 \leq \theta \leq \pi$; b) $\Phi(0) = \Phi(\pi) = 0$.

This natural assumption will play a fundamental role in what follows and will allow us to use the theory of positive operators.

Denote by B the Banach space of functions $\Phi(\theta)$ continuous on the interval $[0;\pi]$ and satisfying conditions b), $\|\Phi\| = \max \Phi$. Obviously, the set of elements of B which satisfy inequality a) forms a cone K in B ([5]).

Introduce the operator A by the relation

$$A\Phi = \int_0^{\pi} K(\varepsilon\theta) e^{3C\Phi} \sin \Phi \, d\varepsilon$$

and formulate the problem finally in the following form: find a function $\Phi \in K,\ \Phi \neq 0$, which satisfies the equation

$$\Phi = \nu A \Phi$$

for some $\nu > 0$.

It is easy to show that to each solution of our problem there corresponds a fluid flow, and the parameters determining the flow are connected by the condition

$$\frac{g\lambda}{2\pi c^2} = \nu.$$

For the operator A the following theorem is valid: The operator A is defined in the ball of the space B of radius $r < \pi/6$, acts in this space, and is completely continuous.

The proof of this theorem follows from the inequality

$$\int_0^{2\pi} e^{\lambda C\Phi} \leqslant \max \frac{2\pi}{\cos \lambda\Phi}, \qquad \text{where } |\lambda\Phi| < \frac{\pi}{2}, \tag{1}$$

and from certain properties of the kernel $K(\varepsilon\theta)$.

Next note that the kernel $K(\varepsilon\theta)$ is nonnegative in the square $0 \leqslant \varepsilon \leqslant \pi;\ 0 \leqslant \theta \leqslant \pi$. Hence it follows that the operator A, defined on the intersection K_r of the ball of radius r of the space B with the cone K, is positive. Further, it is easy to show that the operator A has on K_r a monotone minorant and, consequently, by the theorem of M. A. Krasnosel' skii ([2], p. 269), the positive eigenfunctions of the operator A form in K a continuous branch of length r. By the usual methods of the theory of positive operators it is shown that the positive spectrum of A lies entirely in some finite interval $a < \nu < b$; $a > 0$, $b > 0$. Here a and b do not depend on $\|\Phi\|$, and therefore, even if there exist waves with angle of inclination of the tangent to the profile greater than $\pi/6$, the corresponding Froude numbers belong to $(a; b)$.

From this, in particular, it follows that in a fluid of infinite depth, with bounded velocity, arbitrarily long waves cannot exist. This fact no longer holds for a fluid of finite depth.

Thus, we have proved that there exist periodic symmetric waves in a fluid of infinite depth for which the maximum angle of inclination to the wave profile assumes any value from the interval $(0; \pi/6)$.

In the case of a fluid of finite depth the problem reduces to determining nonzero solutions of the equation

$$\Phi(\theta) = \frac{g\lambda}{2\pi c^2} \int_0^\pi K_1(\varepsilon\theta) e^{3C_1\Phi} \sin\Phi \, d\varepsilon,$$

analogous to the one considered above. Here c is the mean velocity along the bottom; C_1 is the conjugation operator for the annulus, i.e. the function $i\Phi + C_1\Phi$ is the limiting value on the outer circumference of a function analytic in the annulus.

function which takes real values on the inner circle;

$$K_1(\varepsilon\theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - r_0^{2n}}{1 + r_0^{2n}} \frac{\sin n\theta \sin n\varepsilon}{n}.$$

The solution of this problem depends on two dimensionless parameters:

$$\nu = \frac{g\lambda}{2\pi c^2} \quad \text{and} \quad r_0 = e^{-2\pi h/\lambda};$$

h is the mean depth.

Theorem 1. *For each prescribed value of h/λ there exist steady waves whose maximum angle of inclination assumes any value from the interval $(0; \pi/6)$. The Froude numbers then satisfy the inequalities*

$$a < \frac{g\lambda}{2\pi c^2} < b, \qquad a > 0;$$

a, b depend only on the value of h/λ.

2. Let us now consider the more general problem of a periodic flow of a fluid over a wavy bottom. Let the form of the bottom (S) be given by the parametric equation

$$\alpha = \alpha(l),$$

where α is the angle between the tangent to the curve S and the x-axis, and l is arc length. The function $\alpha(l)$ is periodic with period L. We shall assume that $\alpha(l)$ has a first derivative satisfying the Hölder condition.

Problem 1. Determine an irrotational steady motion of a fluid with a free boundary, possessing the same periodicity as the bottom (S). This problem can be reduced to the solution of the system of equations

$$\Phi(\theta) = \nu \int_0^{2\pi} K_1(\varepsilon\theta) e^{3C\Phi} \sin\Phi \, d\varepsilon + \int_0^{2\pi} K_2(\varepsilon\theta)\alpha[l(\varepsilon)] \, d\varepsilon; \qquad l(\theta) = \frac{L}{2\pi}\int_\theta^{2\pi} e^{C\alpha} \, d\varepsilon$$

with unknowns $\Phi(\theta)$ and $l(\theta)$. Here $\Phi(\theta)$, as before, is the angle of inclination of the tangent to the profile of the free boundary; $K_1(\varepsilon\theta)$ and $K_2(\varepsilon\theta)$ are the Green' s functions of the mixed problem for an annulus; C is the conjugation operator. The solution is determined by two dimensionless parameters: the Froude number and the relative depth

$$\nu = \frac{gL}{2\pi c^2}; \qquad r_0 = e^{-2\pi h/L};$$

c is the mean velocity along the bottom; h is the mean depth. The following theorem concerns rapid supercritical flows.

Theorem 2. *Let a function $\alpha(l)$ and a parameter r_0 be given, and suppose that the inequalities*

$$\frac{8r_0}{1-r_0}\max|\alpha| < \frac{\pi}{6}; \qquad \max|\alpha| < \frac{\pi}{2}$$

hold. Then one can specify a positive constant a such that, for every $0 < \nu < a$, the solution of Problem 1 exists and is unique.

Let us note that, when the conditions of Theorem 2 are fulfilled, the solution of Problem 1 depends continuously on the function $\alpha(l)$ and on the parameters r_0 and ν. In particular, the curve of the profile of the free boundary tends to a straight line as the depth increases according to an exponential law. More precisely, the inequality

$$\max|\Phi| < \frac{8e^{-2\pi h/L}}{1-e^{-2\pi h/L}} \max|\alpha|$$

is valid.

In the case when the curve S has vertical axes of symmetry and the angle $\alpha(l)$ preserves its sign over a half-period, one may pose the following problem:

Problem 2. Find periodic flows in which the free boundary has the same vertical axes of symmetry, and the angle $\Phi(\theta)$ has, on a half-period, the same sign as $\alpha(l)$. Following Gerber, we shall call such flows P-flows.

Theorem 3. *Suppose that the following conditions are satisfied:*

1. $$\frac{4r_0}{1-r_0}\max|\alpha| = \alpha_0 < \frac{\pi}{6}, \qquad \max|\alpha| < \frac{\pi}{2}.$$

2. $$\alpha(-l) = -\alpha(l).$$

3. $$\alpha(l) \geq 0, \qquad 0 < l < L/2.$$

Then for every number $\beta \in (\alpha_0; \pi/6)$ there exists a P-flow such that $\max|\Phi| = \beta$. The numbers ν then belong to some interval $(0; b)$, $b > 0$, and b is independent of β.

Flows of type P were studied by Gerber [4]. He proved that for small values of ν such flows exist and are unique. On the other hand, in [3] it was established that for large values of ν there cannot exist P-flows of small amplitude. Using the theory of positive operators, we show that for large values of ν P-flows of any amplitude are impossible.

In conclusion, we note that the number $\pi/6$ in the formulations of Theorems 1 and 3 is apparently not accidental. It is known that limiting Stokes waves (whose existence has not yet been proved) have a maximum angle of inclination exactly $\pi/6$. Moreover, using inequality (1), it is easy to show that the curve of the wave profile will be arbitrarily smooth and, according to Gerber' s results [4], an analytic curve if $\max|\Phi| < \pi/6$. In this connection it would be very interesting to prove that steady waves with $\max|\Phi| > \pi/6$ do not exist at all.

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Note: Figure translations are in progress. See original paper for figures.