MATHEMATICS

1960

SovietRxiv

Abstract

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MATHEMATICS

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ON THE FILTERING OF RANDOM SEQUENCES

(Presented by Academician A. A. Blagonravov, 10 X 1959)

1. Consider a random function $u(t)$, which is measured at discrete instants of time

$$\dots, t_{n+2}, t_{n+1}, t_n$$

(the instant t_{k+1} precedes the instant t_k; t_n is the instant of the last measurement). The measurements contain random errors ξ_k, so that they yield the values

$$v_k = v(t_k) = u(t_k) + \xi_k.$$

In a number of practical problems connected with the use of discrete control machines, it is required, from the measurement results v $(k = n, n+1, n+2, \dots)$, to find the best approximation to the quantity $u_n = u(t_n)$. In doing so, it is essential what amount of memory and how many operations are required for the computations.

As an example of such problems, we point to the determination of the wind velocity u on American bombers: from the results v_k of measurements, at equal time intervals, of the wind velocity u_k, the quantity

$$s_n = \frac{31}{32} s_{n+1} + \frac{1}{32} v,$$

is constantly (and recurrently) computed; this is then taken as the approximate value of u.

The solutions of the problems considered may be different depending on the known properties of the random function $u(t)$.

Below we give solutions for the case of Brownian motion, when $u(t)$ has random increments with mathematical expectation equal to zero (Theorem 1), and for the case when $u(t)$ is the sum of two functions–one having random increments, and the other having random second differences (Theorem 2). Theorem 3 has a negative significance: it shows the impossibility of generalizing Theorem 1 to a case weaker than in Theorem 2; there does not exist a best solution of the problem of determining the position of a Brownian particle in a flow with constant (unknown) velocity.

2.1. Let the values u_i

$$\dots, u_2, u_1, u_0$$

have random increments ζ_i, so that

$$u_i = u_{i+1} + \zeta_i.$$

The values u_i are measured with errors ξ_i, so that the measurement gives the result

$$v_i = u_i + \xi_i.$$

It is assumed that ζ_i and ξ_i are independent random variables with mathematical expectations equal to zero:

$$M\zeta_i = M\xi_i = 0,$$

and constant variances

$$\mathbf{D}^2\zeta_i = D_1^2, \qquad \mathbf{D}^2\xi_i = D_2^2.$$

It is required to find a linear function $f(\{v_i\})$ of the results of the measurements v_i, possessing the following properties:

a) for any fixed (but unknown) value u_0, the mathematical expectation $\mathbf{M}f$ must be equal to u_0:

$$\mathbf{M}f(\{v_i\}) = u_0;$$

b) among all functions possessing property a), the required function f must have minimal variance $\mathbf{D}^2 f$:

$$\mathbf{D}^2 f(\{v_i\}) = \mathbf{M}[f(\{v_i\}) - u_0]^2.$$

2.2. Theorem 1. The solution of the problem posed in 2.1 is the function

$$f = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i v_i, \tag{1}$$

where λ is determined from the equation

$$\frac{\lambda}{(1-\lambda)^2} = \frac{\mathbf{D}^2\xi}{\mathbf{D}^2\zeta}, \qquad \text{i.e. } \lambda = \delta - \sqrt{\delta^2 - 1}; \qquad \delta = 1 + \frac{D_1^2}{2D_2^2}.$$

In this case

$$\mathbf{D}^2 f = (1-\lambda)\mathbf{D}^2\xi = \frac{\lambda}{1-\lambda}\mathbf{D}^2\zeta.$$

Remark 1. It follows from Theorem 1 that the best approximations f to the quantities u_n are found from the known results of previous measurements $\{v_i\}$ $(i = n, n+1, ...)$ by the recurrence formula

$$f = \lambda f_{n+1} + (1-\lambda)v_n,$$

since

$$f_k = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i v_{k+i} \quad \text{(see (1))}.$$

Remark 2. In the case $\mathbf{D}^2\zeta \ll \mathbf{D}^2\xi$, when the function u changes little and the measurements are rough, the number λ is close to 1, and

$$1-\lambda \simeq \sqrt{\mathbf{D}^2\zeta/\mathbf{D}^2\xi}, \qquad \mathbf{D}^2 f \simeq \sqrt{\mathbf{D}^2\zeta\cdot\mathbf{D}^2\xi}.$$

3.1. Let us now consider the case when each u_i,

$$..., u_2, u_1, u_0,$$

is the sum of quantities h_i and k_i,

$$u_i = h_i + k_i$$

where k_i is a quantity with independent random increments (first differences) ζ_i:

$$k_i - k_{i+1} = \zeta_i, \qquad \text{i.e. } k_i = k_{i+1} + \zeta_i,$$

and h_i is a quantity with independent random second differences η_i:

$$h_i - h_{i+1} = z_i, \qquad z_i - z_{i+1} = \eta_i.$$

Suppose the values u_i are measured with error ξ_i, so that the result of measurement is

$$v_i = u_i + \xi_i.$$

Suppose, further, that η_i, ζ_i and ξ_i are independent random variables with mathematical expectation

$$\mathbf{M}\eta_i = \mathbf{M}\zeta_i = \mathbf{M}\xi_i = 0$$

and constant variances

$$\mathbf{D}^2\eta_i = D_1^2, \qquad \mathbf{D}^2\zeta_i = D_2, \qquad \mathbf{D}^2\xi_i = D_3^2.$$

It is required to find a linear function $f_0 = f(\{v_i\})$ of the results of measurements $\{v_i\}$ possessing the following properties:

a$_1$) for any fixed (unknown) values u_0 and u_1, the mathematical expectation of f_0 must be equal to u_0:

$$\mathbf{M}[f(\{v_i\})] = u_0;$$

b$_1$) among all functions possessing property a$_1$), the required function must have the least variance

$$\mathbf{D}^2 f(\{v_i\}) = \mathbf{M}[f(\{v_i\}) - u_0]^2.$$

3.2. Before formulating Theorem 2, which gives the solution of this problem, we shall make some remarks about the quantities D_1^2 and D_3^2 and introduce the necessary notation.

We note that one may assume $D_3^2 \neq 0$, since in the case $D_3^2 = 0$, evidently, $f_0 = v_0 = u_0$. Further, we shall assume $D_1^2 \neq 0$; the case $D_1^2 = 0$ is considered in Theorem 3.

Let

$$d_1 = \frac{D_1^2}{D_3^2}, \qquad d_2 = \frac{D_2^2}{D_3^2}.$$

Next, denote by β_1 and β_2 those two of the four roots of the equation

$$\beta^4 - (4 + d_2)\beta^3 + (6 + 2d_2 + d_1)\beta^2 - (4 + d_2)\beta + 1 = 0,$$

whose moduli are less than one. It is easy to see that under the assumption made, $D_1^2 \neq 0$, such roots β_1 and β_2 always exist.

Theorem 2. *The solution of the problem posed in subsection 3.1, under the conditions* $D_1^2 \neq 0$, $D_3^2 \neq 0$, $d_2^2 \neq 4d_1$, *is the function*

$$f_0 = f(\{v_i\}) = \frac{\beta_1\beta_2}{\beta_2(1-\beta_1)-\beta_1(1-\beta_2)} \sum_{n=0}^{\infty} \left[(1-\beta_1)^2\beta_1^{n-1} - (1-\beta_2)^2\beta_2^{n-1}\right] v_n. \tag{2}$$

In the case $d_2^2 = 4d_1$ the problem has no unique solution.

From Theorem 2 it is easy to obtain recurrence relations for computing the best approximation f_n to the quantity u_n. Namely, in the case of real β_1 and β_2 (which corresponds to $d_2 > 4d_1$), f_n can be obtained as the sum

$$f_n = A_n + B_n,$$

where A_n and B_n are quantities computed recurrently:

$$A_n = \beta_1 A_{n+1} + \delta_1 v_n, \qquad B_n = \beta_2 B_{n+1} + \delta_2 v_n; \tag{3}$$

here

$$\delta_1 = \frac{\beta_2(1-\beta_1)^2}{\beta_2(1-\beta_1)-\beta_1(1-\beta_2)}, \qquad \delta_2 = \frac{\beta_1(1-\beta_2)^2}{\beta_2(1-\beta_1)-\beta_1(1-\beta_2)}.$$

In the case $d_2^2 < 4d_1$, when β_1 and β_2 are complex conjugates, $\beta_1 = a + ib$, $\beta_2 = a - ib$, it is convenient to use formulas in which f_n is computed recurrently simultaneously with the auxiliary quantity g_n:

$$f_n = af_{n+1} - bg_{n+1} + cv_n, \qquad g_n = bf_{n+1} + ag_{n+1} - dv_n; \tag{4}$$

here

$$c = 1 + a - 2(a^2+b^2), \qquad d = b + 2\frac{1-a}{b}(a^2+b^2-a).$$

Since $|\beta_1| < 1$ and $|\beta_2| < 1$, formulas (3), as well as formulas (4), ensure stable computation; computational errors do not accumulate, and the influence of the quantities v_{n+k}, A_{n+k}, and B_{n+k} in formulas (3), respectively v_{n+k}, f_{n+k}, and g_{n+k} in formulas (4), tends to zero as $k \to \infty$.

Theorem 2 admits a generalization to the case when u_i is the sum of a finite number s of quantities with random first, second, ···, s-th differences.

4. Let the values u_i

$$\dots, u_2, u_1, u_0$$

satisfy all the requirements set forth in Sec. 2.1, with the exception of the condition $\mathrm{M}\zeta_i = 0$, which we now replace by the requirement that the mean value of ζ_i be a constant:

$$\mathrm{M}\zeta_i = C.$$

It is essential to bear in mind that the quantity C is assumed by us to be unknown in advance.

Theorem 3. *There does not exist a function*

$$f = \sum_{i=0}^{\infty} a_i v_i$$

of the results $\{v_i\}$ *of measuring the quantities* u_i $(i = 0, 1, 2, \dots)$, *satisfying conditions a) and b) of Sec. 2.1, if* $\mathrm{M}\zeta_i$ *is an unknown constant.*

Let us note that if only a finite number of values v_i $(i = 0, 1, 2, \dots, n)$ is known, then a function

$$f = \sum_{i=0}^{n} a_i v_i,$$

satisfying conditions a) and b) of Sec. 2.1 for $\mathrm{M}\zeta_i = C$, can be found. However, the set of limiting values $\{a_i(n)\}$ as $n \to \infty$ no longer satisfies the conditions of the problem.

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Received
6 IX 1959

Note: Figure translations are in progress. See original paper for figures.