MATHEMATICS

I. V. USHAKOVA

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. V. USHAKOVA

A UNIQUENESS THEOREM FOR FUNCTIONS HOLOMORPHIC AND BOUNDED IN THE UNIT DISK

(Presented by Academician V. I. Smirnov on July 2, 1959)

The classical uniqueness theorem of the theory of functions states:

Let $\{\zeta_k\}$ *be a sequence of points lying inside the unit disk and satisfying the condition*

$$\sum_k 1 - |\zeta_k| = \infty; \tag{A}$$

in this case a function $f(\zeta)$*, holomorphic and bounded in the unit disk, is identically equal to zero if*

$$f(\zeta_k) = 0 \quad (k = 0, 1, 2, \ldots). \tag{B}$$

The present note is devoted to a generalization of this proposition; namely, we shall prove that the following theorem is true.

Theorem. *Let* $\{\zeta_k\}$ $(\zeta_k \to 1)$ *be a sequence of points lying inside a sector with vertex at the point* $\zeta = 1$*, formed by two chords of the circle* $|\zeta| = 1$*, and suppose that, in addition to (A), the condition*

$$\left|\frac{1+\zeta_k}{1-\zeta_k}\right| - \left|\frac{1+\zeta_{k+1}}{1-\zeta_{k-1}}\right| \geq d > 0; \tag{A'}$$

is satisfied; in this case every function $f(\zeta)$*, holomorphic and bounded inside the unit disk, is identically equal to zero if the relation*

$$\lim_{k\to\infty} |1-\zeta_k| \ln |f(\zeta_k)| = -\infty. \tag{B'}$$

is satisfied.

Our conditions on $\{\zeta_k\}$ are stronger than in the classical theorem, but the condition (B$'$) on $f(\zeta)$ is weaker than (B).

Proof. We pass from the unit disk $|\zeta| < 1$ to the half-plane $\operatorname{Im} z > 0$,

$$z = i\frac{1+\zeta}{1-\zeta}, \qquad z_k = i\frac{1+\zeta_k}{1-\zeta}, \tag{1}$$

Then, putting $1-\zeta_k = \rho_k e^{i\theta_k}$, $\cos\delta \le \cos\theta_k \le 1$, so that δ is the half-opening of the angle in which the points ζ_k are situated and which we take to be symmetric with respect to the real axis, we shall have

$$\tfrac{1}{2}(1-|\zeta_k|) \le |z_k|^{-1} \le M_k(1-|\zeta_k|) \quad \left(\lim_{k\to\infty} M_k = \sec\delta\right). \tag{2}$$

It follows from (2) that condition (A) is equivalent to the condition

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|} = \infty, \tag{A_1}$$

and conditions (A') and (B') take, respectively, the form

$$|z_{k+1}| - |z_k| \ge d > 0; \tag{A_1'}$$

$$\lim_{k\to\infty} \frac{\ln|F(z_k)|}{|z_k|} = -\infty, \tag{B_1'}$$

where $F(z) = f(\zeta)$, by virtue of the conditions of our theorem, is bounded for $\operatorname{Im} z > 0$, and if $F(z) \not\equiv 0$, then, by Carleman' s theorem ((1), p. 293), the integral has a finite value

$$\int_{-\infty}^{\infty} \frac{\ln|F(t)|}{1+t^2}\, dt. \tag{3}$$

Assuming our theorem to be false, suppose that $F(z) \not\equiv 0$, and denote the zeros of $F(z)$ by α_k $(\operatorname{Im}\alpha_k > 0)$.

By virtue of the classical theorem cited above, the zeros α_k lying outside the unit circle satisfy the inequality

$$\sum_{k=1}^{\infty} \left|\operatorname{Im}\frac{1}{\alpha_k}\right| < \infty. \tag{4}$$

Indeed, for $\operatorname{Im}\alpha_k > 0,\ |\alpha_k| \ge 1$,

$$\sum_{k=1}^{\infty}\left|\operatorname{Im}\frac{1}{\alpha_k}\right| \leq \sum_{k=1}^{\infty}\left\{1-\left|\frac{\alpha_k-i}{\alpha_k+i}\right|^2\right\} \leq 2\sum_{k=1}^{\infty}\left\{1-\left|\frac{\alpha_k-i}{\alpha_k+i}\right|\right\}.$$

In this case, on the basis of a general theorem of B. Ya. Levin ((1), p. 311), for the function $F(z)$ the following representation is valid:

$$\ln|F(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F(t)|}{(x-t)^2+y^2}\,dt + ky + \ln|\chi(z)|, \tag{5}$$

where

$$k = h_F\left(\frac{\pi}{2}\right), \qquad \chi(z) = \prod_{1}^{\infty}\left(1-\frac{z}{\alpha_k}\right)\left(1-\frac{z}{\overline{\alpha_k}}\right)^{-1}.$$

The Blaschke product $\chi(z)$, by virtue of (4), converges absolutely and uniformly in every finite part of the plane not containing the zeros α_k.

Let us now turn to the points z_k, which, thanks to the condition of the theorem and relation (1), must be situated inside some angle Γ having the imaginary axis as bisector. Denote the aperture of this angle by σ. Therefore $y_k \leq |z_k| \leq y_k \sec\sigma$. Since the integral (3) exists, uniformly inside the angle Γ,

$$\lim_{y\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F(t)|}{(x-t)^2+y^2}\,dt = 0.$$

Hence, by virtue of (5) and (B_1'),

$$\lim_{k\to\infty}\frac{\ln|\chi(z_k)|}{|z_k|} = -\infty,$$

and therefore our problem consists in proving that this relation is incompatible with conditions (A_1) and (A_1') of our theorem.

A related problem was considered by Levinson ((^{2}), p. 107), namely, he obtained the following theorem:

Let $\Phi(z)$ *be an analytic function of exponential type in the upper half-plane, bounded on the real axis. Let* $\{z_n\}$ *be a sequence of complex numbers such that*

$$\lim_{n\to\infty}\frac{n}{z_n} = De^{\pi i/2},$$

and suppose that for the numbers of this sequence there exists a $d > 0$ *such that*

$$|z_n - z_m| \geq |n-m|d.$$

Then, in order that the relation

$$\limsup_{n\to\infty} \frac{\ln|\Phi(z_n)|}{z_n} = \limsup_{y\to\infty} \frac{\ln|\Phi(iy)|}{y},$$

hold, it is necessary and sufficient that the series

$$\sum_{1}^{\infty} \frac{1}{|z_n|}$$

converge.

As Boas (3) observed, the theorem ceases to be true if one does not require that $\arg z_n \to \pi/2$ $(n \to \infty)$.

We could complete the proof with the aid of Levinson' s theorem. However, in Levinson' s theorem the requirements imposed on the sequence of points $\{z_k\}$ are somewhat different from ours. In our case the points $\{z_k\}$ may lie inside an arbitrary angle Γ in the upper half-plane, whereas in Levinson $\arg z_k \to \pi/2$; on the other hand, his condition of noncondensation of the sequence is weaker than ours.

To obtain the required result we shall use certain theorems on the Blaschke product.

1°. The equality

$$\overset{*}{\lim_{r\to\infty}} \frac{\ln|\chi(re^{i\theta})|}{r} = 0 \qquad (0 < \theta < \pi),$$

holds, where the asterisk means that the point $z = re^{i\theta}$ tends to infinity while avoiding some system of circles C_j having zero relative measure, i.e. such that

$$\lim_{R\to\infty} \frac{1}{R} \sum_j R_j = 0,$$

where the summation is over the radii of all those circles C_j which lie in the disk $|z| < R$. This result belongs to B. Ya. Levin ((1), p. 307).

2°. There exists an infinite sequence of intervals $\Delta_n = [a_n, b_n]$ $(0 < a_n < b_n)$, for which

$$\int_\Delta \frac{dr}{r} < \infty, \qquad \lim_{\substack{r\to\infty \\ r\in\Delta}} \frac{1}{r} \ln|\chi(re^{i\theta})| = 0, \qquad \left|\theta - \frac{\pi}{2}\right| < \sigma, \tag{6}$$

where $\Delta = \sum \Delta_n$ is a set of finite logarithmic length.

Thus, here the point $z = re^{i\theta}$ must tend to infinity outside the annular sectors

$$\left|\theta - \frac{\pi}{2}\right| \leq \sigma, \qquad a_n < r < b_n \quad (n = 1, 2, \ldots).$$

This is a result of Ahlfors and Heins (^{6}). The intersection of the totality of these sectors with the set of exceptional circles C_j is some system of domains, which we denote by $\{D_n\}$. Then

$$\lim_{r\to\infty} \frac{\ln|\chi(re^{i\theta})|}{r} = 0 \qquad \left(re^{i\theta} \in \overline{D} = \sum D_n\right). \tag{7}$$

Let us exclude from D those domains D_n which contain no points α_k, and denote the remaining set by D'. Then (7) will hold if r tends to infinity in such a way that $re^{i\theta} \in D'$. Indeed, if a given domain D_n contains no points α_k, then $\ln|\chi(z)|$ is a harmonic function in D_n. By (7), on the boundary of the domain D_n, for any arbitrarily small $\varepsilon > 0$ and for $r \leq r_0(\varepsilon)$, the inequality

$$\ln|\chi(re^{i\theta})| > -\varepsilon r$$

will be satisfied. By the minimum principle this inequality (possibly with a larger ε)

will hold everywhere in D_n. But in that case $\lim \frac{1}{r}\ln|\chi(re^{i\theta})| \geq -\varepsilon$ ($r \to \infty$, $re^{i\theta} \in D'$), and hence (7) is satisfied for $re^{i\theta} \in D'$. Consequently, in each domain $D_n \subset D'$ there lies at least one of the points α_n. Let α'_n be some zero of $\chi(z)$ which has fallen into such a domain D_n. Starting with some n, all the domains D_n containing points of the sequence $\{z_k\}$ lie inside the angle Γ', $|\arg z - \pi/2| \leq \sigma' < \pi/2$, which contains Γ. Therefore, starting with some n, one will have $|\alpha'_n| \leq \sec\sigma' \cdot \operatorname{Im}\alpha_n$, and consequently

$$\sum_1^\infty \frac{1}{|\alpha'_n|} < \infty.$$

Take the half-ring corresponding to the interval Δ_n, and let the points $z_{k_n}, z_{k_n+1}, \dots, z_{k_n+p-1}$ lie in it. Then $a_n < |z_n| < b_n < e^c a_n$; $a_n < |\alpha'_n| < b_n < e^c a_n$, where $c = \int_\Delta \frac{dr}{r}$, whence it follows that

$$\frac{1}{|z_{k_n}|} < e^c \frac{1}{|\alpha'_n|}.$$

Therefore

$$\int_{\Delta_n} \frac{dr}{r} > d\sum_{i=0}^{p-1} \frac{1}{|z_{k_n+i}|} - \frac{d}{|z_{k_n}|} \geq d\left(\sum_{i=0}^{p-1} \frac{1}{|z_{k_n+i}|} - \frac{e^c}{|\alpha'_n|}\right),$$

and, taking condition (A_1) into account, we find that

$$\frac{1}{|z_{k_n}|} < e^c \frac{1}{|\alpha'_n|}, \qquad \int_\Delta \frac{dr}{r} \geq d\left(\sum_{k=1}^\infty \frac{1}{|z_k|} - e^c \sum_{n=1}^\infty \frac{1}{|\alpha'_n|}\right) = \infty,$$

and this contradicts inequality (6). Thus the theorem is proved.

It carries over without difficulty to functions of the class $N(A)$ (see (4)), for which $\ln|f(\zeta)|$ has a harmonic majorant. Indeed, such a function is represented, by a well-known theorem of R. Nevanlinna (5), in the form of the quotient of two bounded functions $f_1(\zeta), f_2(\zeta)$, the denominator having no zeros in the disk $|\zeta| < 1$. After passing to the half-plane z, we obtain the representation

$$\ln|F_1(z)| = -\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F_1(t)|}{y^2+(x-t)^2}\,dt - \ln|\chi_1(z)| - \ln|F_2(z)| + ky, \quad k = h_{F_1}\left(\frac{\pi}{2}\right).$$

Since $\ln|F_2(z)|$ is a harmonic function for $\operatorname{Im} z > 0$, we have

$$\ln|F_2(z)| = -\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F_2(t)|}{y^2+(x-t)^2}\,dt,$$

and, consequently,

$$\lim_{k\to\infty}\frac{\ln|F_2(z_k)|}{|z_k|} = 0.$$

Thus the problem is reduced to the one considered by us.

I express my sincere gratitude to Prof. B. Ya. Levin for suggesting the topic and for consultation.

Kharkov State University
named after A. M. Gorky

Received
30 VI 1959

CITED LITERATURE

1. B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956.
2. N. Levinson, *Gap and Density Theorems*, 1940.
3. R. Boas, *Studia Math.*, **13**, 59 (1953).
4. I. I. Privalov, *Boundary Properties of Analytic Functions*, Moscow, 1950.
5. R. Nevanlinna, *Math. Ann.*, **92**, 145 (1924).
6. L. Alfors, M. Heins, *Ann. of Math.* (2), **50**, 341 (1949).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.