



Soviet-era science, translated into English

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1960

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Abstract

Full Text

MATHEMATICS

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THE SIXTH MOMENT FOR L -SERIES AND AN ASYMPTOTIC FORMULA IN THE HARDY-LITTLEWOOD PROBLEM

The Hardy-Littlewood equation ^(1,2)

$$n = p + \xi^2 + \eta^2 \quad (1)$$

(p is a prime number, ξ, η are integers) has a solution for all sufficiently large n . For the number $Q(n)$ of solutions of equation (1), the following asymptotic formula holds*:

Theorem 1.

$$Q(n) \sim \pi \frac{n}{\ln n} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)} \right) \prod_{p|n} \frac{\chi^2(p-1)(p - \chi_4(p))}{p^2 - p + \chi_4(p)} + R(n), \quad (2)$$

where

$$R(n) = O\left(\frac{n}{(\ln n)^{1.028}}\right). \quad (3)$$

Equation (1), as was indicated in ⁽²⁾, reduces to equations of the form

$$x_1 x_2 \dots x_k + \xi^2 + \eta^2 = n, \quad (4)$$

where x_i ($i = 1, 2, \dots, k$) run through numbers with sufficiently large prime factors. For $k > 6$ the solutions of such equations have been obtained by the "dispersion method" ⁽³⁾; for $k \leq 6$ they reduce to equations of the form (4), where the x_i run successively through all odd numbers. The solution of all such equations (for $k \leq 6$) with the proper asymptotic for the number of solutions can be carried out with the aid of the following theorem on the "sixth moment for L -series."

Let D run through all integers subject to the condition

$$D_1 \leq D \leq D_1 + D_2, \quad (5)$$

where $D_2 \gg D_1 (\ln D_1)^{-k}$; $k > 0$ is some constant. For each modulus D consider all Dirichlet characters χ_D and the L -series $L(s, \chi_D)$; let $s = \frac{1}{2} + it$ (t real). Form the sixth (unnormalized) moment for L -series:

$$\sum_{D_1 \leq D \leq D_1 + D_2} \sum_{\chi_D} \left| L\left(\frac{1}{2} + it, \chi_D\right) \right|^6. \quad (6)$$

The solution of equations of the form (4) for $k \leq 6$ is achieved with the help of an estimate of the sixth (unnormalized) moment (6). The following theorem holds, for which other applications may also be indicated.

* In my note (2) a gap was admitted in the reasoning, as a result of which the remainder term (3) comes out worse than indicated in (2).

Theorem 2.

$$\sum_{D_1 \leq D \leq D_1 + D_2} \sum_{\chi_D} \left| L\left(\frac{1}{2} + it, \chi_D\right) \right|^6 = BD_2 D_1 (|t| + 1)^{l_0} \exp(\ln D_1)^{\varepsilon_0}, \quad (7)$$

where B is a bounded quantity, $l_0 > 0$ is a constant; $\varepsilon_0 > 0$ is an arbitrarily small constant.

Estimate (7) for the sixth moment of L -series makes it possible to derive certain new results in the additive divisor problem, and also to obtain rather good information on the number of solutions of the equation

$$n = p + \varphi(x, y), \quad (8)$$

where $\varphi(x, y)$ is an arbitrary integral binary quadratic form.

Received
9 V 1960

REFERENCES

¹ G. H. Hardy, J. E. Littlewood, Acta Math., 44, 1 (1923). ² Yu. V. Linnik, DAN, 124, No. 1, 29 (1959). ³ Yu. V. Linnik, DAN, 120, No. 5, 960 (1958).

Note: Figure translations are in progress. See original paper for figures.

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