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Abstract

Full Text

MATHEMATICS

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ON A CLASS OF SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

(Presented by Academician N. I. Muskhelishvili on 6 I 1960)

Let L be a closed smooth contour of Lyapunov type in the plane of the complex variable z ; let S^+ be the finite domain bounded by the contour L . Denote by S^- the domain which complements $S^+ + L$ to the full plane. We shall assume that the point $z = 0$ belongs to the domain S^+ . Consider the singular integro-differential equation

$$K\varphi \equiv \sum_{r=0}^m \left[A_r(t_0)\varphi^{(r)}(t_0) + \frac{1}{\pi i} \int_L \frac{K_r(t_0, t)\varphi^{(r)}(t) dt}{t - t_0} \right] = f(t_0), \quad (1)$$

where $A_r(t)$, $K_r(t_0, t)$, $f(t)$ are functions prescribed on L , and $A_m(t_0) \pm K_m(t_0, t_0)$ do not vanish on L ; $\varphi(t)$ is the unknown function.

In the present note, under the assumption that the prescribed functions satisfy certain smoothness conditions, theorems are established which are analogues of the well-known Noether theorems (see, for example, ⁽¹⁾).

There exist various methods of reducing equation (1) to a singular integral equation (for a bibliography on this question see ⁽²⁾). One of the methods of studying equation (1) is its reduction to a differential boundary-value problem*.

Represent the function $K_m(t_0, t)$ in the form

$$K_m(t_0, t) = B_m(t_0) + (t - t_0)k_m(t_0, t), \quad \text{where} \quad B_m(t_0) = K_m(t_0, t_0).$$

In what follows we shall assume that there exist Hölder-continuous derivatives

$$\frac{d^r A_r(t)}{dt^r}, \quad \frac{\partial^r K_r(t_0, t)}{\partial t_0^j \partial t^{r-j}} \quad (r = 0, 1, \dots, m; j = 0, 1, \dots, r), \quad \frac{\partial^m k_m(t_0, t)}{\partial t^m}.$$

It is easy to see that the derivative $\varphi^{(m)}(t)$ of every solution of equation (1) belongs to the class H , i.e. satisfies the Hölder condition.

Consider the piecewise holomorphic function $\Phi(z)$ associated with the solution of equation (1) by the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t) dt}{t - z}.$$

Obviously, $\Phi(z)$ will be a solution vanishing at infinity of the following differential boundary-value problem:

$$\sum_{r=0}^m \left\{ A_r(t_0) [\Phi^{+(r)}(t_0) - \Phi^{-(r)}(t_0)] + \frac{1}{\pi i} \int_L \frac{K_r(t_0, t) [\Phi^{+(r)}(t) - \Phi^{-(r)}(t)] dt}{t - t_0} \right\} = f(t_0), \quad (2)$$

* We use the terms employed in (1,3).

where $\Phi^{(r)+}(t)$ and $\Phi^{(r)-}(t)$ are the boundary values of the derivative $\Phi^{(r)}(z)$, respectively from S^+ and S^- .

With the aid of solutions of problem (2) that vanish at infinity, the solutions of equation (1) are obtained by the formula $\varphi(t) = \Phi^+(t) - \Phi^-(t)$.

On the basis of the results of paper (3), the necessary and sufficient conditions for solvability of equation (1) have the form

$$\int_L f(t) \psi_j(t) dt = 0, \quad j = 1, 2, \dots, k',$$

where $\psi_1(t), \psi_2(t), \dots, \psi_{k'}(t)$ is a complete system of linearly independent solutions of an equation of the form

$$\begin{aligned} & \frac{A_m(t_0)}{t_0^m} \psi(t_0) - \frac{1}{\pi i} \int_L \frac{B_m(t) \psi(t) dt}{t_0^m (t - t_0)} + \\ & + \sum_{r=0}^{m-1} \int_L q_r(t, t_0) (t - t_0)^{m-r-1} \psi(t) \ln(t - t_0) dt + \\ & + \sum_{r=0}^{m-1} \frac{1}{2\pi i} \int_L \omega_r(t_1, t_0) (t_1 - t_0)^{m-r-1} \ln(t_1 - t_0) dt_1 \frac{1}{\pi i} \int_L \frac{K_r(t, t_1) \psi(t) dt}{t_1 - t} + \\ & + \frac{1}{2\pi i} \int_L \omega_m(t_1, t_0) \ln(t_1 - t_0) dt_1 \frac{1}{\pi i} \int_L \frac{K_m(t, t_1) \psi(t) dt}{t_1 - t} + \int_L P(t, t_0) \psi(t) dt = 0, \end{aligned} \quad (3)$$

where $q_r(t, t_0)$, $\omega_r(t, t_0)$, $P(t, t_0)$ are certain functions independent of $\psi(t)$. Under our assumptions there exist derivatives of class H

$$\frac{\partial^{r+m} q_r(t, t_0)}{\partial t^r \partial t_0^m} \quad (r = 0, 1, \dots, m-1); \quad \frac{\partial^{r+m} \omega_r(t, t_0)}{\partial t^r \partial t_0^m} \quad (r = 0, 1, \dots, m); \quad \frac{\partial^m P(t, t_0)}{\partial t_0^m},$$

The following propositions hold:

1°. Let $K(t_0, t)$ be a function of class H having an integrable bounded derivative $K'_{t_0}(t_0, t)$. Then

$$\begin{aligned} & \frac{d}{dt_0} \int_L K(t_0, t) \ln(t - t_0) dt = \\ & = \int_L K'_{t_0}(t_0, t) \ln(t - t_0) dt - \int_L \frac{K(t_0, t) dt}{t - t_0} + \pi i K(t_0, t_0), \end{aligned} \quad (4)$$

2°. Let the function $K(t_0, t)$ have n derivatives of class H

$$\frac{\partial^r K(t_0, t)}{\partial t_0^j \partial t^{r-j}}, \quad j = 0, 1, \dots, r.$$

Then

$$\frac{d^r}{dt_0^r} \int_L \frac{K(t_0, t) dt}{t - t_0} = \int_L \frac{(\partial/\partial t_0 + \partial/\partial t)^r K(t_0, t)}{t - t_0} dt. \quad (5)$$

On the basis of the expression for the solutions of an equation of the form (see ⁽¹⁾, p. 134)

$$A(t_0)\psi(t_0) - \frac{1}{\pi i} \int_L \frac{B(t)\psi(t) dt}{t - t_0} = g(t_0) \quad (6)$$

and on the basis of formula (5), it is easy to see that, if the given functions $A(t)$, $B(t)$, and $g(t)$ have derivatives of class H of order r , then every solution of equation (6) has a derivative of order r of class H .

On the basis of what was set out above, using formulas (4) and (5), we conclude that every solution of equation (3) has a derivative of order m of class H .

As was proved in (3), equation (3) is equivalent to the infinite system

$$\int_L \omega_j^{(1)}(t)\psi(t) dt = 0, \quad (7')$$

$$\int_L \omega_j^{(2)}(t)\psi(t) dt = 0, \quad j = 0, 1, 2, \dots \quad (7'')$$

In the case considered by us, $\omega_j^{(1)}(t) = Kt^j$, $\omega_j^{(2)}(t) = Kt^{-j-1}$, where the operator K is given by formula (1).

From system (7') it follows that

$$\int_L t^j \left[A_m(t)\psi(t) + \frac{1}{\pi i} \int_L \frac{K_m(t_1, t)\psi(t_1) dt_1}{t - t_1} - V_{m-1}(t) \right] dt = 0, \quad j = 0, 1, 2, \dots, \quad (8)$$

$$\int_L \left[A_r(t)\psi(t) + \frac{1}{\pi i} \int_L \frac{K_r(t_1, t)\psi(t_1) dt_1}{t - t_1} - V_{r-1}(t) \right] dt = 0, \quad r = 0, 1, \dots, m,$$

where the functions $V_r(t)$ are defined by the formula

$$V_r(t_0) = \int_c^{t_0} \left[A_r(t)\psi(t) + \frac{1}{\pi i} \int_L \frac{K_r(t_1, t)\psi(t_1) dt_1}{t - t_1} - V_{r-1}(t) \right] dt, \quad (9)$$

$$r = 0, 1, \dots, m-1, \quad V_{-1}(t) \equiv 0,$$

where c is a fixed point on L .

Condition (8) means that

$$A_m(t)\psi(t) + \frac{1}{\pi i} \int_L \frac{K_m(t_1, t)\psi(t_1) dt_1}{t - t_1} - V_{m-1}(t) = \Psi(t), \quad (10)$$

where $\Psi(t)$ is the boundary value on L of a function $\Psi(z)$ holomorphic in S^+ .

From system (7'') we derive:

$$\int_L \left[A_m(t)\psi(t) + \frac{1}{\pi i} \int_L \frac{K_m(t_1, t)\psi(t_1) dt_1}{t - t_1} - V_{m-1}(t) \right] t^{-j-m-1} dt = 0;$$

hence $\Psi^{(m)}(t) \equiv 0$.

By virtue of (5), (9), and (10), the last relation is reduced to the equation

$$K'\psi \equiv \sum_{r=0}^m (-1)^r \left\{ [A_r(t)\psi(t)]^{(r)} - \frac{1}{\pi i} \int_L \frac{(\partial/\partial t + \partial/\partial t_1)^r K_r(t_1, t)\psi(t_1)}{t_1 - t} dt_1 \right\} = 0. \quad (11)$$

Thus, every solution of equation (3) is a solution of the integro-differential equation (11). The converse assertion is also true. Consequently, equations (3) and (11) are equivalent.

We shall call equation (11) adjoint to equation (1). It can be shown that equation (1) is also adjoint to equation (11).

From what has been set out, the validity of the following theorem follows.

Theorem 1. The necessary and sufficient conditions for solvability of the singular integro-differential equation $K\varphi = f$ are that

$$\int_L f(t)\psi_j(t) dt = 0, \quad j = 1, 2, \dots, k',$$

where $\psi_1(t), \psi_2(t), \dots, \psi_{k'}(t)$ is a complete system of linearly independent solutions of the adjoint homogeneous equation $K'\psi = 0$.

According to (3'), we have $k - k' = \varkappa$, where k and k' are the numbers of linearly independent solutions of equations (1) and (3), and \varkappa is given by the formula

$$\varkappa = \frac{1}{2\pi} \left\{ \arg \frac{A_m(t) - B_m(t)}{A_m(t) + B_m(t)} \right\}_L.$$

We shall call the number \varkappa the index of the operator K . Obviously, the following holds:

Theorem 2. *The difference between the number of linearly independent solutions of the homogeneous equation $K\varphi = 0$ and the number of linearly independent solutions of the adjoint homogeneous equation $K'\psi = 0$ is equal to the index of the operator K .*

The theorems stated remain valid also for systems of integro-differential equations of the form (1). In this case the adjoint equation is defined by the formula

$$K'\psi \equiv \sum_{r=0}^m (-1)^r \left\{ [A'_r(t)\psi(t)]^{(r)} - \frac{1}{\pi i} \int_L \frac{(\partial/\partial t + \partial/\partial t_1)^r K'_r(t_1, t)\psi(t_1)}{t_1 - t} dt_1 \right\} = 0,$$

where $A'_r(t)$ and $K'_r(t, t_0)$ are matrices obtained from the matrices $A_r(t)$ and $K_r(t, t_0)$ by transposing their elements.

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REFERENCES

- ¹ N. I. Muskhelishvili, *Singular Integral Equations*, M.-L., 1946.
- ² N. P. Vekua, Reports of the Academy of Sciences of the Georgian SSR, 22, No. 1, 3 (1959).
- ³ R. S. Isakhanov, Reports of the Academy of Sciences of the Georgian SSR, 20, No. 6, 659 (1958).

Note: Figure translations are in progress. See original paper for figures.

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