

FIXED POINTS OF OPERATORS THAT COMPRESS OR STRETCH A CONE

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Abstract

Full Text

MATHEMATICS

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FIXED POINTS OF OPERATORS THAT COMPRESS OR STRETCH A CONE

(Presented by Academician P. S. Aleksandrov, 15 VI 1960)

1. Below we study operators that leave invariant a cone K in a real Banach space E . As usual ^(1,2), we shall write $x \preceq y$ if $y - x \in K$; if $y - x \in \overset{\circ}{K}$, then we shall write $x \prec y$.

Theorem 1. *Let the operator A be completely continuous, and suppose that there exists an $R > 0$ such that*

$$Ax \succ x \quad (x \in K, \|x\| \geq R). \quad (1)$$

Then the operator A has at least one fixed point in the cone K .

2. In the following assertions it is assumed that $A\theta = \theta$, where θ is the zero of the space E . The question considered is whether there exists in the cone K a second (nonzero) fixed point.

We shall say that the operator A ($A\theta = \theta$) **compresses** the cone K if there exist positive R and r such that condition (1) and the condition

$$Ax \preceq x \quad (x \in K, \|x\| \leq r) \quad (2)$$

are fulfilled.

The operator A ($A\theta = \theta$) **stretches** the cone if there exist such R and r that

$$Ax \preceq x \quad (x \in K, \|x\| \geq R); \quad (3)$$

$$Ax \succ x \quad (x \in K, \|x\| \leq r). \quad (4)$$

Theorem 2. *Let the completely continuous operator A compress or stretch the cone K . Then A has in the cone K at least one nonzero fixed point.*

Theorems 1 and 2 are proved by topological methods.

If the cone K is solid (this assumption is not present in the hypotheses of the theorems), then one can apply the theory of completely continuous vector

fields ⁽²⁾. In this case denote by T the set of those elements $x \in K$ for which $r \leq \|x\| \leq R$. Denote the boundary of the domain T by Γ . If the vector field $x - Ax$ has no zero vectors on Γ , then its rotation is equal to 1 if A compresses the cone, and is equal to -1 if A stretches the cone.

Conditions (1) or (4) are usually checked without difficulty. For example, in a number of cases one can introduce in the space E a norm $\|x\|_0$ that is monotone (from $x \preceq y$, where $x, y \in K$, it follows that $\|x\|_0 \leq \|y\|_0$), equivalent to the old norm, and chosen in such a way that on elements $x \in K$ with, respectively, small or large norm the inequality $\|Ax\|_0 < \|x\|_0$ is satisfied. Verification of conditions (2) or (3) along this path can be carried out only in exceptional cases. If, for example, the completely continuous operator A is linear, and the cone K does not admit plastering (see ⁽³⁾), then the condition $\|Ax\| \geq \|x\|$ ($x \in K$) certainly cannot be fulfilled. Recall that the cones of nonnegative functions in the space C and in the spaces L_p , where $p > 1$, do not admit plastering.

3. We shall call an operator A_- (respectively, A_+) a **minorant (majorant)** of the operator A on the set T , if $A_-x \ll Ax$ ($A_+x \gg Ax$) for $x \in T$. It is clear that from $A_-x \ll x$ there follows the relation $Ax \ll x$, and from $A_+x \gg x$ the relation $Ax \gg x$. Therefore majorants and minorants can be used in checking the conditions of Theorems 1 and 2.

As minorants and majorants one may choose linear operators B . Conditions (1)–(4) can then be checked by relying on Theorems 3 and 4 given below.

A linear operator B leaving the cone K invariant is called **u_0 -bounded from below (from above)** if to each nonzero $x \in K$ there correspond a natural number $p = p(x)$ and a positive number $a = a(x)$ such that $B^p x \gg au_0$ ($B^p x \ll au_0$). Here u_0 denotes some nonzero element of K .

Theorem 3. *Let u_0 be a linear operator B bounded from below and satisfying the condition $Bu_0 \gg (1 + \varepsilon_0)u_0$, where $\varepsilon_0 > 0$.*

Then $Bx \ll x$ for all $x \in K$, $x \neq \theta$.

Theorem 4. *Let u_0 be a linear operator B bounded from above and satisfying the condition $Bu_0 \ll (1 - \varepsilon_0)u_0$, where $\varepsilon_0 > 0$.*

Then $Bx \gg x$ for all $x \in K$, $x \neq \theta$.

4. In applications of Theorem 1 and, especially, Theorem 2 to the study of concrete operator equations, it is sometimes useful first to redefine the operator A on elements of large and small norm in such a way that the fixed points of the redefined operator \tilde{A} coincide with the fixed points of the operator A , and so that Theorem 1 or 2 is directly applicable to the operator \tilde{A} . Such a redefinition is facilitated if the operator A is differentiable at the point θ and at infinity.

Let $A'(\theta)$ and $A'(\infty)$ be linear operators such that

$$\lim_{x \in K, \|x\| \rightarrow 0} \frac{\|Ax - A\theta - A'(\theta)x\|}{\|x\|} = \lim_{x \in K, \|x\| \rightarrow \infty} \frac{\|Ax - A'(\infty)x\|}{\|x\|} = 0.$$

Theorem 5. *Let A ($A\theta = \theta$) be completely continuous. Suppose each of the operators $A'(\theta)$ and $A'(\infty)$ has in the cone K a unique eigenvector; denote the corresponding eigenvalues by $\lambda(\theta)$ and $\lambda(\infty)$.*

Then, for the existence of a nonzero fixed point of the operator A in the cone K , it is sufficient that the condition $1 \in (\lambda(\theta), \lambda(\infty))$ be fulfilled.

We emphasize that in the hypotheses of Theorem 5 either the inequality $\lambda(\theta) < \lambda(\infty)$ or the inequality $\lambda(\theta) > \lambda(\infty)$ may hold.

The conditions for the existence of a nonzero fixed point may be formulated in a mixed form: if $\lambda(\theta) < 1$, then it is sufficient that condition (3) be satisfied; if $\lambda(\theta) > 1$, condition (1); if $\lambda(\infty) < 1$, condition (2); finally, if $\lambda(\infty) > 1$, it is sufficient that condition (4) be satisfied.

5. We shall say that **condition (a)** is fulfilled if the space E is weakly complete, the unit ball in E is weakly compact, and the cone K admits plastering. The weak topology may be introduced not by means of all functionals from the conjugate space E^* , but by means of some subspace $E_0^* \subset E^*$; it is then only necessary to ensure that in E_0^* one can find such a functional $f(x)$ that $f(x) \geq \|x\|$ for $x \in K$.

Theorem 6. *Let condition (a) be fulfilled. Let the operator A be weakly continuous and satisfy condition (1). Then A has at least one fixed point on K .*

In this theorem (as in Theorem 1), condition (1) may be replaced by the assumption of the existence of the operator $A'(\infty)$ and the inequality $\lambda(\infty) < 1$.

Theorem 7. *Theorems 2 and 5 remain valid if the assumption of complete continuity of the operator A is replaced by condition (a) and the assumption of weak continuity of the operator A .*

6. By the usual schemes (see (2)), one obtains applications of the above-mentioned new fixed-point principles to nonlinear integral equations or nonlinear boundary-value problems (for equations involving operators whose Green functions are nonnegative). As examples of applications we shall indicate some simple, but apparently new, theorems on the existence of periodic solutions for systems of ordinary differential equations.

For the proof of the assertions below concerning first-order systems, a certain cone is singled out in the phase space and fixed points are sought for the operator defined by shifting along the trajectories of the system through a time equal to the period; that is, the usual Poincaré principle of point transformations is applied. The study of systems of second-order equations is carried out by another method: the problem is reduced to an integral equation, and then fixed

points of the integral operator are sought in a certain cone singled out in the corresponding space of vector-functions.

7. Consider the system

$$\dot{x}_i = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \quad (5)$$

where the right-hand sides are assumed to be periodic in t with period ω . For simplicity we shall assume that the functions f_i are defined and continuously differentiable with respect to the totality of variables for $-\infty < t, x_j < \infty$ (this assumption also applies to the subsequent examples).

Suppose that system (5) has the zero solution, i.e. $f_i(t, 0, \dots, 0) \equiv 0$ ($i = 1, \dots, n$), and that

$$f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \geq 0 \quad (x_j \geq 0, j \neq i).$$

Then the functions $a_{ij} = D_{x_j} f_i(t, 0, \dots, 0)$ for $i \neq j$ will be nonnegative; we shall assume that they are positive. Then the monodromy matrix of the system $\dot{x}_i = a_{i1}(t)x_1 + \dots + a_{in}(t)x_n$ has a positive eigenvalue λ_0 , which is greater than the remaining eigenvalues (in modulus).

Assume further that $W(x) = W(x_1, \dots, x_n)$ is a positive-definite quadratic form such that either, for nonnegative x_1, \dots, x_n and sufficiently large values of the sum $x_1 + \dots + x_n$, the inequality

$$\sum_1^n W'_{x_i} f_i(t, x_1, \dots, x_n) \geq 0, \quad (6)$$

holds, or, for the indicated values of the variables,

$$\sum_1^n W'_{x_i} f_i(t, x_1, \dots, x_n) \leq 0. \quad (7)$$

Then it follows from Theorem 2 that *system (5) has at least one nonzero periodic solution, provided either condition (6) and the inequality $\lambda_0 < 1$ hold, or condition (7) and the inequality $\lambda_0 > 1$ hold.*

8. As a second example, consider the equation

$$\ddot{x} + f(t, x, \dot{x}) = 0. \quad (8)$$

Let f be periodic in t and satisfy one of the conditions:

1°. $f(t, x, y) \equiv -f(-t, -x, y)$.

$$2^\circ. \quad f(t, x, y) \equiv -f\left(\frac{\omega}{2} + t, -x, -y\right).$$

Further, let $f(t, x, y) \geq 0$ for $0 \leq t \leq \omega/2$ and $x \geq 0$, and suppose that for $x \geq 0$

$$\lim_{x+|y| \rightarrow 0} \sup_t \frac{|f(t, x, y) - a_0(t)x|}{x + |y|} = \lim_{x+|y| \rightarrow \infty} \sup_t \frac{|f(t, x, y) - a_\infty(t)x|}{x + |y|} = 0.$$

It follows from Theorem 5 that, for equation (8) to have a nonzero periodic solution, it is sufficient that one of the differential equations

$$\ddot{x} + a_0(t)x = 0, \quad \ddot{x} + a_\infty(t)x = 0$$

have no solutions (different from the trivial one) that assume the value zero twice on $[0, \omega/2]$, and that the second equation have a nontrivial solution assuming the value zero twice on the interval $(0, \omega/2)$. This theorem can also be obtained by more elementary considerations, which cease to be applicable in passing to systems; at the same time Theorem 5 immediately leads to analogous assertions for systems of equations of the form (8).

9. As a final example, consider the system

$$\ddot{x}_i = -f_i(x_1, \dots, x_n; \dot{x}_1, \dots, \dot{x}_n) \quad (i = 1, \dots, n). \quad (9)$$

Suppose that the right-hand sides are odd either jointly in the first n variables or jointly in all $2n$ variables. Suppose that, for $x_j \geq 0$ and all values of y_k , the inequalities

$$\begin{aligned} m^2(x_1 + \dots + x_n) &\leq f_i(x_1, \dots, x_n; y_1, \dots, y_n) \leq \\ &\leq m_1^2(x_1 + \dots + x_n) + M \quad (i = 1, \dots, n) \end{aligned}$$

hold. Finally, suppose that there exists a $\delta_0 > 0$ such that, for $0 < x_j < \delta_0$, $|y_k| < \delta_0$,

$$f_i(x_1, \dots, x_n; y_1, \dots, y_n) \geq m_0^2(x_1 + \dots + x_n) \quad (i = 1, \dots, n).$$

It follows from Theorem 2 that the autonomous system (9) has nonzero periodic solutions with a continuum of different periods, if $m_1 < m_0$.

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