A METHOD FOR THE NUMERICAL SOLUTION AND NUMERICAL ANALYSIS OF SOLUTIONS OF HOMOGENEOUS SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS OF GENERAL FORM

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Abstract

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A METHOD FOR THE NUMERICAL SOLUTION AND NUMERICAL ANALYSIS OF SOLUTIONS OF HOMOGENEOUS SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS OF GENERAL FORM

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The numerical solution of homogeneous systems of linear algebraic equations which, in matrix notation, have the form

$$(A - \lambda B)X = 0, \tag{1}$$

where A and B are matrices of order n; X is a column matrix with n elements; λ is a number, is especially difficult when both matrices A and B are singular. The method set forth below is suitable both in the case of nonsingular matrices A and B, and in cases where one of them, or both simultaneously, are singular.

The method consists in the successive application of the following operations:

1) numerical solution of equation (1) for some one value of the root of the secular equation

$$|A - \lambda B| = 0; \tag{2}$$

2) passage to the equation

$$(A^{(1)} - \lambda B^{(1)})X' = 0 \tag{3}$$

with matrices $A^{(1)}$ and $B^{(1)}$ of order $(n-1)$, which has solutions for the same values of λ as equation (1);

3) passage from the numerical solution of equation (3), corresponding to a root of the secular equation

$$|A^{(1)} - \lambda B^{(1)}| = 0, \tag{4}$$

to the corresponding solution of equation (1).

The reduction of the order of the matrices in equation (1) is carried out until it becomes equal to 2. For the numerical solution of equation (1) for some one value of the root of equation (2), one may use the author' s iterative method ([1,2]).

The reduction of the order of the matrices in equations of the form (1) is based on the following considerations. The equation

$$|PAR - \lambda PBR| = 0, \tag{5}$$

where A and B are the matrices of equation (1), and P and R are arbitrary nonsingular matrices of order n, has the same roots as equation (2). Further, the solution $X^{(i)'}$ of the equation

$$(PAR - \lambda PBR)X' = 0 \tag{6}$$

is related to the corresponding solution $X^{(i)}$ of equation (1) by the relation

$$X^{(i)} = RX^{(i)'}. \tag{7}$$

We shall assume, without loss of generality, that $X^{(0)} = (1, x_2^{(0)}, \dots, x_n^{(0)})$ is one of the solutions of equation (1) corresponding to the root λ_0 of equation (2). Then, assuming that $BX^{(0)} \neq 0$ (this assumption is always fulfilled if $|B| \neq 0$) and choosing the matrices R and P in the form

$$R = \left\| \begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \\ x_2^{(0)} & 1 & 0 & \dots & 0 \\ x_3^{(0)} & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(0)} & 0 & 0 & \dots & 1 \end{array} \right\|, \qquad P = \left\| \begin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \\ -z_2^{(0)} & 1 & 0 & \dots & 0 \\ -z_3^{(0)} & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -z_n^{(0)} & 0 & 0 & \dots & 1 \end{array} \right\|,$$

where $z_i^{(0)}$ $(i = 2, 3, \dots, n)$ are the elements of the column matrix $Z^{(0)} = (1, z_2^{(0)}, \dots, z_n^{(0)})$, determined by the relation

$$BX^{(0)} = Z^{(0)} \cdot (BX^{(0)})_1, \tag{8}$$

$(BX^{(0)})_1$ being the first element of $BX^{(0)}$, we obtain the system of equations corresponding to equation (6) in the form

$$\begin{aligned}(\lambda_0-\lambda)(BX^{(0)})_1\cdot x_1'+(a_{12}-\lambda b_{12})x_2'+\ldots+(a_{1n}-\lambda b_{1n})x_n'&=0,\\ 0\cdot x_1'+(a_{22}^{(1)}-\lambda b_{22}^{(1)})x_2'+\ldots+(a_{2n}^{(1)}-\lambda b_{2n}^{(1)})x_n'&=0,\\ \ldots&\\ \ldots&\\ 0\cdot x_1'+(a_{n2}^{(1)}-\lambda b_{n2}^{(1)})x_2'+\ldots+(a_{nn}^{(1)}-\lambda b_{nn}^{(1)})x_n'&=0,\end{aligned} \tag{9}$$

where

$$a_{ij}^{(1)}=(APR)_{ij}=a_{ij}-z_i^{(0)}a_{1j},\qquad b_{ij}^{(1)}=(PBR)_{ij}=b_{ij}-z_i^{(0)}b_{1j} \tag{10}$$

$$(i,j=2,3,\ldots,n).$$

Since the roots of equations (5) and (2) coincide, the remaining roots of equation (2) are the roots of equation (4), in which the elements of the matrices $A^{(1)}=\|a_{ij}^{(1)}\|$ and $B^{(1)}=\|b_{ij}^{(1)}\|$ are determined by relations (10).

If $X_0^{(i)'}=(x_2^{(i)'},x_3^{(i)'},\ldots,x_n^{(i)'})$ is a solution of equation (3) corresponding to the root λ_i of equations (4), (5), and (2), then $X^{(i)'}=(x_1^{(i)'},x_2^{(i)'},\ldots,x_n^{(i)'})$, with

$$x_1^{(i)'}=\frac{\sum_{j=2}^{n}(a_{1j}-\lambda_i b_{1j})x_j^{(i)'}}{(\lambda_i-\lambda_0)(BX^{(0)})_1} \tag{11}$$

is the corresponding solution of equation (6). To obtain the solution $X^{(i)}$ of equation (1) corresponding to the root λ_i of equation (2), it remains only to use formula (7), which gives, in expanded form,

$$x_1^{(i)}=x_1^{(i)'};\qquad x_j^{(i)}=x_j^{(i)'}+x_j^{(0)}\cdot x_1^{(i)}\quad (j=2,3,\ldots,n). \tag{12}$$

If λ_0 is an l-fold root of equation (2) and all m linearly independent solutions $X^{(0)},X^{(1)},\ldots,X^{(m-1)}$ of equation (1) corresponding to it are known, then, upon passing to equation (3) by means of, for example, $X^{(0)}$, one should also find all $m-1$ linearly independent solutions of equation (3) corresponding to this root, using the obvious relation

$$X^{(i)'}=R^{-1}X^{(i)}\qquad (i=1,2,\ldots,m-1) \tag{13}$$

and discarding the first element in each of the matrices $X^{(i)'}$.

It follows from the form of the matrix R that

$$x_j^{(i)'} = x_j^{(i)} - x_j^{(0)} \cdot x_1^{(i)} \quad (i = 1, 2, \dots, m-1;\ j = 2, 3, \dots, n). \tag{14}$$

With the aid of one of the solutions of equation (3) thus obtained, corresponding to λ_0, one can lower the order of the matrices $A^{(1)}$ and $B^{(1)}$ and, continuing this process, ultimately arrive at an equation of the form (1) with matrices A and B of order $n-m$.

If $m = l$, then the resulting equation of the form (2) of degree $n-l$ no longer has the root λ_0. If, however, $m < l$, then the solutions obtained subsequently of the "shortened" equations of the form (3), corresponding to the root λ_0, beginning with the solution of the equation with matrices of order $n-m$, must obviously be discarded after they have been used to lower the degree of the secular equation.

If the rank of A is $r < n$, but $|B| \neq 0$, then $\lambda_0 = 0$ is at least an $(n-r)$-fold root of equation (2), which makes it possible, by means of the process described above, to lower the order of the matrices A and B by $n-r$. If the rank of B is $r < n$, but $|A| \neq 0$, then by the substitution $\lambda = 1/\mu$ this case is reduced to the preceding one.

If $|A| = 0$ and $|B| = 0$, then three cases may arise.

1. The matrix $X^{(0)}$, satisfying the equation

$$AX = 0, \tag{15}$$

does not satisfy the equation

$$BX = 0. \tag{16}$$

2. The matrix $X^{(0)}$ satisfies equations (15) and (16), but the matrix $Y^{(0)}$, satisfying the equation with the transposed matrix

$$\widetilde{A}Y = 0, \tag{17}$$

does not satisfy the equation

$$\widetilde{B}Y = 0. \tag{18}$$

3. The matrix $X^{(0)}$ satisfies equations (15) and (16), and the matrix $Y^{(0)}$ satisfies equations (17) and (18).

In the first case $X^{(0)}$ satisfies equation (1) for $\lambda = 0$, and since $BX^{(0)} \neq 0$, the order of the matrices A and B can be lowered by the method indicated above.

In the second case $Y^{(0)}$ satisfies the equation

$$(\widetilde{A} - \lambda \widetilde{B})Y = 0 \tag{19}$$

for $\lambda = 0$. Since in this case $\widetilde{B}Y^{(0)} \neq 0$, the order of the matrices $\widetilde{A}$ and $\widetilde{B}$ can be lowered by using the former method.

In the third case, in order to lower the order of the matrices A and B, it is necessary only to delete, for example, the first row and the first column of these matrices. To obtain a solution of equation (1) from the solution of the shortened equation constructed in this way, it is necessary in this case only to supplement the latter (solution) with the value $x_1 = 0$.

If r is the smaller of the ranks of A and B, then, using the considerations set forth above, one can without difficulty lower the order of the matrices A and B by $n - r$.

The equation determining the derivatives of any k-th order of λ_i and $X^{(i)}$, satisfying equation (1), with respect to arbitrary parameters $t_1, t_2, \dots, t_k$,

on which the matrices A and B depend (provided only that these matrices have all the required derivatives with respect to these parameters), has the form

$$(A - \lambda_i B)\frac{\partial^k X^{(i)}}{\partial t_1\,\partial t_2 \cdots \partial t_k} - BX^{(i)}\frac{\partial^k \lambda_i}{\partial t_1\,\partial t_2 \cdots \partial t_k} = -f^{(i)}_{(t_1,t_2,\dots,t_k)}, \tag{20}$$

where

$$f^{(i)}_{(t_1,\dots,t_k)} = (f^{(i)}_{(t_1,\dots,t_k)1}, f^{(i)}_{(t_1,\dots,t_k)2}, \dots, f^{(i)}_{(t_1,\dots,t_k)n})$$

is a column matrix determined by the recurrence relations

$$\begin{aligned} f^{(i)}_{(t_1,\dots,t_k)} = \frac{\partial f^{(i)}_{(t_1,\dots,t_{k-1})}}{\partial t_k} &+ \left(\frac{\partial A}{\partial t_k} - \lambda_i \frac{\partial B}{\partial t_k}\right)\frac{\partial^{k-1} X^{(i)}}{\partial t_1 \cdots \partial t_{k-1}} \\ &- B\left(\frac{\partial^{k-1} X^{(i)}}{\partial t_1 \cdots \partial t_{k-1}}\frac{\partial \lambda_i}{\partial t_k} + \frac{\partial X^{(i)}}{\partial t_k}\frac{\partial^{k-1}\lambda_i}{\partial t_1 \cdots \partial t_{k-1}}\right) \\ &- \frac{\partial B}{\partial t_k} X^{(i)} \frac{\partial^{k-1}\lambda_i}{\partial t_1 \cdots \partial t_k}; \end{aligned} \tag{21}$$

$$f^{(i)}_{(t_1)} = \left(\frac{\partial A}{\partial t_1} - \lambda^{(i)}\frac{\partial B}{\partial t_1}\right) X^{(i)}.$$

Multiplying (20) on the left by the row matrix $\widetilde{Y}^{(i)}$, transposed to the matrix $Y^{(i)} = (1, y_2^{(i)}, \dots, y_n^{(i)})$, satisfying equation (19) for $\lambda = \lambda_i$, we obtain the following convenient formula for computing only $\partial^k \lambda_i / \partial t_1 \cdots \partial t_k$ in the case when λ_i is a finite simple root of equation (2):

$$\frac{\partial^k \lambda_i}{\partial t_1 \cdots \partial t_k} = -\frac{\widetilde{Y}^{(i)} f^{(i)}_{(t_1,\dots,t_k)}}{\widetilde{Y}^{(i)} B X^{(i)}}. \tag{22}$$

For computing $\partial^k \lambda_i / \partial t_1 \cdots \partial t_k$ and $\partial^k X^{(i)} / \partial t_1 \cdots \partial t_k$ in this case, it is more convenient to use the formulas

$$\frac{\partial^k \lambda_i}{\partial t_1 \cdots \partial t_k} = \frac{f^{(i)}_{(t_1,\dots,t_k)1} - V^{(i)\prime} W^{(i)-1} f^{(i)0}_{(t_1,\dots,t_k)}}{B_1 X^{(i)} - V^{(i)\prime} W^{(i)-1} B_2 X^{(i)}}, \tag{23}$$

$$\frac{\partial^k X^{(i)0}}{\partial t_1 \cdots \partial t_k} = -W^{(i)-1} f^{(i)0}_{(t_1,\dots,t_k)} + W^{(i)-1} B_2 X^{(i)} \frac{\partial^k \lambda_i}{\partial t_1 \cdots \partial t_k}, \tag{24}$$

which are obtained from equation (19) under the assumption that $\partial^k x_1^{(i)} / \partial t_1 \cdots \partial t_k = 0$.

In formulas (23) and (24), B_1 is the first row of the matrix B; B_2 is the rectangular matrix obtained by deleting the first row of the matrix B; $f^{(i)0}_{(t_1,\dots,t_k)}$ is a column matrix with elements $f^{(i)}_{(t_1,\dots,t_k)j}$ $(j = 2, \dots, n)$; $V^{(i)\prime} = a' - \lambda_i b'$ is a row matrix with elements $v_j^{(i)} = a_{1j} - \lambda_i b_{1j}$ $(j = 2, 3, \dots, n)$; $W^{(i)} = A_0 - \lambda_i B_0$ is a matrix of order $(n-1)$ with elements $w_{kj}^{(i)} = a_{kj} - \lambda_i b_{kj}$ $(k, j = 2, 3, \dots, n)$.

When solving equation (1) by the author' s iteration method (1,2), the computational scheme of the last iteration step that led to the obtaining of λ_i and $X^{(i)0} = (x_2^{(i)}, x_3^{(i)}, \dots, x_n^{(i)})$ makes it possible without difficulty to compute both $Y^{(i)0} = (y_2^{(i)}, y_3^{(i)}, \dots, y_n^{(i)})$ (3), and $W^{(i)-1} f^{(i)0}_{(t_1,\dots,t_k)}$ and $W^{(i)-1} B_2 X^{(i)}$. The computation of the derivatives of λ_i and $X^{(i)}$ can be carried out on the basis of the same reduced equation that served to compute λ_i.

All the formulas written above and the computations based on them are correspondingly simplified if the matrices A and B are symmetric or if $B = E$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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