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Abstract

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ANALYTIC SEMIGROUPS AND ILL-POSED PROBLEMS FOR EVOLUTION EQUATIONS

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1°. Let A be a closed unbounded operator that is the generating operator of a strongly continuous semigroup of bounded operators $U(t)$ acting in a Banach space E (^{1}).

For the equation

$$\frac{dx}{dt} = -Ax \tag{1}$$

the problem of finding a solution satisfying the initial condition

$$x(0) = x_0, \tag{2}$$

is ill-posed. However, if one restricts oneself in advance to a certain class of solutions, then in this class the problem may become well-posed. The case when the operator A is a self-adjoint operator in a Hilbert space was considered by one of the authors in (^{2}). In the present paper a broader class of operators acting in a Banach space is considered.

In what follows, by a solution of equation (1) on the interval $[0, T]$ we shall mean a function $x(t)$, continuous in the norm of the space E, having a strong derivative on $(0, T)$, and satisfying equation (1).

Definition. We shall say that problem (1)–(2) is **well-posed in the class of bounded solutions on the interval** $[0, T]$ if, for any M, ε and $\tau \in (0, T)$, there exists a $\delta(M, \varepsilon, \tau)$ such that for every solution $x(t)$ satisfying the conditions

$$\|x(t)\| \le M, \qquad t \in [0, T], \qquad \|x(0)\| \le \delta, \tag{3}$$

the inequality

$$\|x(\tau)\| \le \varepsilon \tag{4}$$

holds.

It is evident that the well-posedness of the problem in the class of bounded solutions implies the uniqueness of its solution.

Every solution $x(t)$ of problem (1)—(2) on $[0,T]$, by the formula $y(t)=x(T-t)$, generates a solution of the problem

$$\frac{dy}{dt}=Ay, \qquad y(0)=x(T). \tag{5}$$

To prove the well-posedness of problem (1)—(2) in the class of bounded solutions, it suffices to obtain an estimate for the solutions $y(t)=U(t)y_0$ of problem (5) in terms of their values at the endpoint $t=T$ and in terms of the maximum of their norm on $[0,T]$. Such estimates are obtained below, under the assumption of analyticity of the semigroup $U(t)$.

2°. Theorem 1. *Let $U(t)$ be a strongly continuous semigroup of bounded operators admitting an analytic continuation into some conical half-neighborhood K in the complex plane (z). Let G be a domain lying together with its closure in K. Denote*

$$N=\max_{z\in\overline{G}}\|U(z)\|.$$

Then for any two points z_0 and z_1 belonging to G, and $y\in E$, the inequality

$$\|U(z_1)y\|\le N^{1-\omega}C^{\omega}\|U(z_0)y\|^{\omega}\|y\|^{1-\omega}, \tag{6}$$

holds, where $C(z_0)$ and $\omega(z_0,z_1)$ are nonnegative functions independent of the choice of y in E.

The proof of Theorem 1 rests on the following lemma.

Lemma. If a function $f(z)$ with values in a Banach space E is analytic in z in a domain G, then $\|f(z)\|$ is a logarithmically subharmonic function in G.

We give the outline of the proof of Theorem 1. From the point z_0 draw a straight line parallel to the real axis. Let z' be the point of intersection of this line with the boundary of the domain G, nearest to z_0 and lying to the right of z_0. Denote by G' the domain obtained from G by making a cut along the segment $[z_0,z']$. For a point z of this segment we have $\|U(z)y\|\le\|U(z-z_0)\|\,\|U(z_0)y\|$. The point $z-z_0$ runs over a segment of the real axis from 0 to $z'-z_0$. Denote by $C(z_0)$ the maximum of the norm of the operator $U(t)$ on this segment. Then on the segment $[z_0,z']$, $\|U(z)y\|\le C(z_0)\|U(z_0)y\|$. On the remaining part of the boundary, $\|U(z)y\|\le\|U(z)\|\,\|y\|\le N\|y\|$. Thus we know an estimate for the logarithmically subharmonic function $\|U(z)y\|$ on the portion $z'z_0z'$ of the boundary of the domain G' and on its complement.

If by $\omega(z_0, z)$ we denote the harmonic measure, constructed for the domain G' with respect to the portion $z' z_0 z'$ of the boundary, then for any logarithmically subharmonic function $\varphi(z)$ in the domain G' Nevanlinna’ s inequality[3] holds:

$$\varphi(z) \leq M^{1-\omega} m^{\omega}, \tag{7}$$

where m is the maximum of the function $\varphi(z)$ on the portion $z' z_0 z'$, and M is the maximum of the function on the remaining part of the boundary. Applying inequality (7) to the function $\|U(z)y\|$, we obtain inequality (6).

From Theorem 1 it follows directly:

Theorem 2. Let A be the infinitesimal generator of a strongly continuous semigroup of bounded operators on $[0, \infty)$, analytic in some conical half-neighborhood K. Then problem (1)–(2) is well posed in the class of bounded solutions on any segment $[0, T]$.

For the proof, for any τ from $(0, T)$ construct a domain G_τ lying, together with its closure, in K and containing inside it the segment $[T - \tau, T]$ of the real axis. Then from inequality (6), applied to the class of solutions of problem (5), corresponding to the class of solutions of problem (1)–(2) bounded by the constant M, there will follow the inequality

$$\|x(\tau)\| = \|y(T-\tau)\| \leq (MN)^{1-\omega} C^{\omega} \|y(T)\|^{\omega} = (MN)^{1-\omega} C^{\omega} \|x(0)\|^{\omega}. \tag{8}$$

If we set

$$\delta = (MN)^{\frac{\omega-1}{\omega}} C^{-1} \varepsilon^{\frac{1}{\omega}}, \tag{9}$$

then (4) follows from (3) and (8). The theorem is proved.

We note that as $T - \tau \to 0$ the boundary of the domain G_τ approaches the boundary of the domain of analyticity of the semigroup $U(z)$, and, consequently, the constant $N = \sup\limits_{z \in \overline{G}_\tau} \|U(z)\|$ may grow without bound. Thus, formula (9) does not clarify the character of the dependence of δ on τ.

3°. Theorem 3. Let the strongly continuous semigroup $U(z)$ be analytic in the sector $-\frac{\pi}{2}\alpha < \arg z < \frac{\pi}{2}\alpha$ $(0 < \alpha < 1)$ and bounded by the constant N_T in the triangle $D_T:\ -\frac{\pi}{2}\alpha \leq \arg z \leq \frac{\pi}{2}\alpha,\ 0 \leq \tau \leq 2T$ $(z = \tau + is)$.

Then for every $y \in E$ the inequality

$$\|U(t)y\| \leqslant N_T \|y\|^{1-\left(\frac{t}{R}\right)^{2/\alpha}} \|U(T)y\|^{\left(\frac{t}{R}\right)^{2/\alpha}}, \tag{10}$$

holds, where

$$R = 2T \cos\frac{\pi\alpha}{4}.$$

Fig. 1

Figure 1: Fig. 1

Proof. Consider the domain F_T bounded by the segments of the straight lines

$$s = \pm\tau \tan\frac{\pi\alpha}{4} \quad \text{and} \quad s = \pm(T-\tau)\tan\frac{\pi}{2}\alpha$$

(see Fig. 1). Obviously, $F_T \subset D_T$. Therefore on the segments OA and OB

$$\|U(z)y\| \leqslant N_T\|y\|.$$

On the segments AC and BC

$$\|U(z)y\| \leqslant \|U(z-T)\|\|U(T)y\| \leqslant N_T\|U(T)y\|,$$

since $z-T \in D_T$. Applying Carleman' s lemma (4) (see also (5)) to the function $U(z)y$ in the domain F_T, we arrive at inequality (10). The theorem is proved.

Fig. 1

Corollary. If the operator A is the infinitesimal generator of a semigroup satisfying the conditions of Theorem 3, then for solutions of problem (1)–(2) inequality (4) follows from inequality (3) for

$$\delta(\varepsilon, M, \tau) = (\varepsilon N_T^{-1})^{\left(\frac{R}{T-\tau}\right)^{2/\alpha}} M^{1-\left(\frac{R}{T-\tau}\right)^{2/\alpha}}.$$

4°. If, under the conditions of Theorem 3, the semigroup $U(z)$ is bounded in the whole sector

$$-\frac{\pi}{2}\alpha < \arg z < \frac{\pi}{2}\alpha,$$

then in inequality (10) the constant N_T may be replaced by a constant independent of T. If t is fixed and T is varied, then inequality (10) makes it possible to estimate from below the possible rate of decay at infinity of solutions of problem (5).

Theorem 4. Let the strongly continuous semigroup $U(z)$ be analytic in the sector

$$-\frac{\pi}{2}\alpha < \arg z < \frac{\pi}{2}\alpha$$

and bounded in it.

Then for every solution $y(t)$ of problem (5), for ε and $t_0 > 0$ there exists a constant β such that

$$\|y(t)\| \geqslant e^{-\beta t^{(1+\varepsilon)/\alpha}}\|y(0)\| \quad (t \geqslant t_0).$$

One class of semigroups satisfying the conditions of Theorem 4 was considered in the works of M. Z. Solomyak ($^{6-8}$). Semigroups of this class are characterized by the following property of the infinitesimal generators A: the resolvent set of the operator $-A$ contains a certain sector

$$\varphi \leqslant \arg\lambda \leqslant 2\pi - \varphi \quad (\varphi < \pi/2),$$

and for any λ from this sector the inequality

$$\|R(\lambda, A)\| = \|(A + \lambda I)^{-1}\| \leqslant \frac{C}{|\lambda| + 1}. \tag{11}$$

holds.

As M. Z. Solomyak showed, inequality (11) holds in the norm of the space $\mathcal{L}_p$ ($p > 1$) for strongly elliptic differential operators under boundary conditions of the first boundary-value problem. From this result and Theorem 3 it follows:

Theorem 5. Let Ω be a bounded domain in n-dimensional space with sufficiently smooth boundary Γ. Let $\mathcal{L}$ be a strongly elliptic differential expression of order $2m$ with sufficiently smooth coefficients*.

* For restrictions on the boundary of the domain and the coefficients of $\mathcal{L}$, see (^{9}), Theorems 9.1–9.6.

For the system of equations

$$\frac{\partial u}{\partial t} = -\mathcal{L}u \tag{12}$$

the problem with boundary conditions

$$u|_\Gamma = \left.\frac{\partial u}{\partial n}\right|_\Gamma = \ldots = \left.\frac{\partial^{m-1} u}{\partial n^{m-1}}\right|_\Gamma = 0 \tag{13}$$

is well posed on every interval $[0, T]$ ($T > 0$) in the class of bounded solutions in $\mathcal{L}_p$.

Using a result of P. E. Sobolevskii, one can show that the assertion of Theorem 5 remains valid in the case where the system (12) consists of a single equation with a second-order elliptic operator under the boundary conditions of the second and third boundary-value problems.

From Theorem 4 it follows:

Theorem 6. For every solution of the equation

$$\frac{\partial u}{\partial t} = \mathcal{L}u,$$

where $\mathcal{L}$ is an elliptic operator with real coefficients satisfying the boundary conditions (13), and for $p > 1$ and $\varepsilon > 0$, there exists a constant β such that

$$\|u(t,x)\|_{\mathcal{L}p} \geq e^{-\beta t^{1+\varepsilon}} \|u(0,x)\|_{\mathcal{L}p}.$$

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Note: Figure translations are in progress. See original paper for figures.