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Abstract

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ON THE GENERAL THEORY OF AXISYMMETRIC MOTIONS OF A GAS

(Presented by Academician L. I. Sedov, 1 IV 1960)

In the present note a class of axisymmetric unsteady adiabatic motions of a perfect gas is considered, whose characteristics admit representations in the form of power series in positive powers of a linear coordinate.

The system of equations of gas dynamics may be written, using a spherical coordinate system, in the form

$$r\frac{\partial \rho}{\partial t} + 2\rho v_r + r\frac{\partial \rho v_r}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) = 0,$$

$$r\frac{\partial s}{\partial t} + r v_r \frac{\partial s}{\partial r} + v_\theta \frac{\partial s}{\partial \theta} = 0, \qquad s = p\rho^{-\gamma},$$

$$\rho r\frac{\partial v_r}{\partial t} + \rho v_r r\frac{\partial v_r}{\partial r} + \rho v_\theta \frac{\partial v_r}{\partial \theta} - \rho v_\theta^2 + r\frac{\partial p}{\partial r} = 0, \tag{1}$$

$$\rho r\frac{\partial v_\theta}{\partial t} + \rho v_r r\frac{\partial v_\theta}{\partial r} + \rho v_\theta \frac{\partial v_\theta}{\partial \theta} + \rho v_r v_\theta + \frac{\partial p}{\partial \theta} = 0,$$

where ρ is the density; p is the pressure; v_r and v_θ are the radial and transverse components of the velocity; γ is the adiabatic exponent; r is the radius; θ is the latitude, measured from the axis of symmetry; t is the time.

In the time derivatives in (1) there is a factor r, whereas in the derivatives with respect to θ there is no such factor. This circumstance makes it possible to find a solution of the system (1) in the form of series in positive powers of r with nonzero constant terms, since for the coefficients of such series one obtains systems of equations containing derivatives with respect to θ and not containing derivatives with respect to time. For this reason the determination of the coefficients reduces to solving ordinary differential equations with coefficients depending on time as on a parameter. The arbitrary constants in the general solution of such systems should evidently be regarded as arbitrary functions of time.

Let us consider a solution of the system (1) having the form:

$$v_r = f_0(t,\theta) + f_1(t,\theta)r + f_2(t,\theta)r^2 + \ldots + f_n(t,\theta)r^n + \ldots ,$$

$$v_\theta = q_0(t,\theta) + q_1(t,\theta)r + q_2(t,\theta)r^2 + \ldots + q_n(t,\theta)r^n + \ldots , \tag{2}$$

$$p = h_0(t) + h_1(t,\theta)r + h_2(t,\theta)r^2 + \ldots + h_n(t,\theta)r^n + \ldots ,$$

$$\rho = g_0(t) + g_1(t,\theta)r + g_2(t,\theta)r^2 + \ldots + g_n(t,\theta)r^n + \ldots .$$

We shall indicate the systems obtained for the coefficients of these expansions, and give an algorithm that makes it possible to find their general solution.

For the zeroth coefficients we have the system

$$g_0(t)\left[2f_0(t,\theta) + \frac{1}{\sin\theta}\frac{\partial q_0(t,\theta)\sin\theta}{\partial\theta}\right] = 0,$$

$$g_0(t)q_0(t,\theta)\left[\frac{\partial f_0(t,\theta)}{\partial\theta} - q_0(t,\theta)\right] = 0, \tag{3}$$

$$g_0(t)q_0(t,\theta)\left[\frac{\partial q_0(t,\theta)}{\partial\theta} + f_0(t,\theta)\right] = 0.$$

Let us consider the case when $f_0(t,\theta)$ and $q_0(t,\theta)$ are not identically equal to zero, thereby assuming that the origin of the coordinate system is not a critical point of the flow. Assuming that $g_0(t) \not\equiv 0$ and $h_0(t) \not\equiv 0$, we shall have the following solution of system (3):

$$\begin{aligned} f_0(t,\theta) &= V_0(t)\cos\theta, & h_0(t) &= h_0(t),\\ q_0(t,\theta) &= -V_0(t)\sin\theta, & g_0(t) &= g_0(t), \end{aligned} \tag{4}$$

where $V_0(t)$, $h_0(t)$, and $g_0(t)$ are arbitrary functions of time. The expressions for $f_0(t,\theta)$ and $q_0(t,\theta)$ reflect the fact that the velocity at the origin of the coordinates is directed along the axis of symmetry, while its magnitude may be an arbitrary function of time.

For the n-th coefficients of the expansions one obtains the system

$$
\begin{aligned}
&(2+n)g_0(t)f_n(t,\theta) + nV_0(t)\cos\theta\, g_n(t,\theta) + \operatorname{ctg}\theta\, g_0(t)q_n(t,\theta) + \\
&\quad + g_0(t)\frac{\partial q_n(t,\theta)}{\partial\theta} - V_0(t)\sin\theta\,\frac{\partial g_n(t,\theta)}{\partial\theta} = \varphi_{n1}\left(\frac{\partial g_{n-1}}{\partial t}, f_k, g_k, q_k, \frac{\partial g_k}{\partial\theta}, \frac{\partial q_k}{\partial\theta}, \theta\right), \\
&n g_0(t)V_0(t)\cos\theta\, h_n(t,\theta) - n\gamma h_0(t)V_0(t)\cos\theta\, g_n(t,\theta) - \\
&\quad - V_0(t)g_0(t)\sin\theta\,\frac{\partial h_n(t,\theta)}{\partial\theta} + \gamma V_0(t)h_0(t)\sin\theta\,\frac{\partial g_n(t,\theta)}{\partial\theta} = \\
&\quad = \varphi_{n2}\left(\frac{\partial h_k}{\partial t}, \frac{\partial g_k}{\partial t}, \frac{\partial h_k}{\partial\theta}, \frac{\partial g_k}{\partial\theta}, f_k, q_k, g_k, h_k\right), \\
&nV_0(t)g_0(t)\cos\theta\, f_n(t,\theta) + nh_n(t,\theta) + \\
&\quad + V_0(t)g_0(t)\sin\theta\, q_n(t,\theta) - V_0(t)g_0(t)\sin\theta\,\frac{\partial f_n(t,\theta)}{\partial\theta} = \\
&\quad = \varphi_{n3}\left(\frac{\partial f_k}{\partial t}, \frac{\partial f_k}{\partial\theta}, g_{k-1}, f_k, q_k\right), \\
&nV_0(t)g_0(t)\cos\theta\, q_n(t,\theta) - V_0(t)g_0(t)\sin\theta\, f_n(t,\theta) + \frac{\partial h_n(t,\theta)}{\partial\theta} - \\
&\quad - V_0(t)g_0(t)\sin\theta\,\frac{\partial q_n(t,\theta)}{\partial\theta} = \varphi_{n4}\left(\frac{\partial q_k}{\partial t}, \frac{\partial q_k}{\partial\theta}, g_{k-1}, f_k, q_k\right),
\end{aligned}
\tag{5}
$$

where $n > 0$, $k = 0, 1, 2, \dots, n-2, n-1$; f_k, q_k, g_k, h_k are the coefficients of the expansions at powers with exponent less than n.

After replacing the unknown functions according to the formulas

$$
g_n(t,\theta) = g_0(t)\bar{g}_n(t,\theta), \qquad h_n(t,\theta) = g_0(t)V_0(t)\bar{h}_n(t,\theta), \tag{6}
$$

we obtain, subsequently omitting the arguments of the functions, the system

$$
\begin{aligned}
&(2+n)f_n + n\cos\theta\,\bar{g}_n - \sin\theta\frac{\partial\bar{g}_n}{\partial\theta} + q_n\operatorname{ctg}\theta + \frac{\partial q_n}{\partial\theta} = \frac{\varphi_{n1}}{g_0}, \\
&nM_0^2\cos\theta\,\bar{h}_n - n\cos\theta\,\bar{g}_n - M_0^2\sin\theta\frac{\partial\bar{h}_n}{\partial\theta} + \sin\theta\frac{\partial\bar{g}_n}{\partial\theta} = \frac{\varphi_{n2}}{g_0}, \\
&n\cos\theta f_n + n\bar{h}_n + \sin\theta q_n - \sin\theta\frac{\partial f_n}{\partial\theta} = \frac{\varphi_{n3}}{V_0 g_0}\cos\theta, \\
&\sin\theta f_n - \frac{\partial\bar{h}_n}{\partial\theta} - n\cos\theta q_n + \sin\theta\frac{\partial q_n}{\partial\theta} = \frac{\varphi_{n4}}{V_0 g_0}\sin\theta,
\end{aligned}
\tag{7}
$$

where $M_0^2 = M_0^2(t) = g_0(t)V_0^2(t)/\gamma h_0(t)$ is the square of the Mach number of the flow at the origin of coordinates. System (7) is a linear nonhomogeneous system of 4 ordinary first-order equations. To obtain the solution of system (7), it is sufficient to find the general solution of the corresponding homogeneous system. Let us find such a solution. Denote it by $f_n^*, q_n^*, g_n^*, h_n^*$.

Integrating the second equation of the homogeneous system, we find:

$$\overline{g}_n^* = M_0^2 \overline{h}_n^* - C_0(t) \sin^n \theta, \tag{8}$$

where $C_0(t)$ is an arbitrary function of time.

Expressing f_n^* from the fourth equation of the homogeneous system and substituting it into the first and third equations, we shall have:

$$(2+n)\frac{\partial \overline{h}_n^*}{\partial \theta} - (1+n)\sin^{s+1}\theta \frac{\partial \overline{q}_n^*}{\partial \theta} + n \sin\theta\cos\theta\, \overline{g}_n^* - \sin^2\theta \frac{\partial \overline{g}_n^*}{\partial \theta} = 0, \tag{9}$$

$$(n+1)\frac{\partial \overline{h}_n^*}{\partial \theta}\operatorname{ctg}\theta - \frac{\partial^2 \overline{h}_n^*}{\partial \theta^2} + n\overline{h}_n^* + \sin^{s+1}\theta \frac{\partial^2 \overline{q}_n^*}{\partial \theta^2} + 2\sin^s\theta\cos\theta (s-n)\frac{\partial \overline{q}_n^*}{\partial \theta} = 0; \tag{10}$$

here

$$\overline{q}_n^* = q_n^* \sin^{-s}\theta, \qquad s = \frac{n(2+n)+1}{1+n}. \tag{11}$$

Eliminating $\overline{q}_n^*$ from (9) and (10), and expressing $\overline{g}_n^*$ in terms of $\overline{h}_n^*$ with the aid of (8), we obtain for $\overline{h}_n^*$ the equation

$$\begin{aligned} &(1 - M_0^2 \sin^2\theta)\frac{\partial^2 \overline{h}_n^*}{\partial \theta^2} + \left[1 + 2(n-1)M_0^2 \sin^2\theta\right] \operatorname{ctg}\theta \frac{\partial \overline{h}_n^*}{\partial \theta} \\ &+ \ n\left[n + 1 - (n-1)M_0^2 + M_0^2 (n-2)\sin^2\theta\right] \overline{h}_n^* = 0. \end{aligned} \tag{12}$$

The general solution of a linear equation of the second order can be found if at least one nontrivial solution of this equation is known. Equation (12) has the following particular solution:

$$\begin{aligned} &\text{for } n \text{ even} \qquad \overline{h}_n^{*0} = \sum_{k=0}^{n/2} \frac{a_{2k}}{a_{20}} \cos^{2k}\theta, \qquad a_{2k} = a_{2k}(M_0, n, a_{20}), \\ &\text{for } n \text{ odd} \qquad \overline{h}_n^{*0} = \sum_{k=0}^{(n-1)/2} \frac{a_{2k+1}}{a_1} \cos^{2k+1}\theta, \quad a_{2k+1} = a_{2k+1}(M_0, n, a_1), \end{aligned} \tag{13}$$

where k takes integer values, and the coefficients a_{2k} and a_{2k+1} are found from algebraic relations.

The general solution of equation (12) has the form:

$$\overline{h}^*_n = c_1(t)\overline{h}^{*0}_n + c_2(t)\overline{h}^{*0}_n \int \frac{(1-M_0^2\sin^2\theta)^{(2n-1)/2}\,d\theta}{\sin\theta\,(\overline{h}^{*0}_n)^2}, \tag{14}$$

where $c_1(t)$ and $c_2(t)$ are arbitrary functions of time.

From (8) we find $\overline{g}^*_n$. Substituting $\overline{h}^*_n$ and $\overline{g}^*_n$ into (9), by one quadrature we obtain $\overline{q}^*_n$, and then, from the fourth equation of the homogeneous system, f^*_n.

Thus, the general solution of the homogeneous system has been found. A particular solution of the nonhomogeneous system (7) is sought in the usual way; after which, taking (6) and (11) into account, we find g_n, h_n, q_n, and f_n.

Let us obtain, as an example, the first coefficients of the expansions. If $n = 1$, then, after substituting (13) into (12), we obtain that $h_1^{*0} = \cos\theta$; hence

$$\bar{h}_1^* = c_1(t)\cos\theta + c_2(t)\cos\theta\left[\sqrt{1+(1-M_0^2)\,\mathrm{tg}^2\,\theta} + \frac{1}{2}\ln\frac{\sqrt{1+(1-M_0^2)\,\mathrm{tg}^2\,\theta}-1}{\sqrt{1+(1-M_0^2)\,\mathrm{tg}^2\,\theta}+1}\right].$$

The second term tends to infinity for $\theta = 0$ and $\theta = \pi$, i.e., on the axis of symmetry, and can obviously be used in problems whose formulation permits a singularity for the pressure and density on the axis of symmetry.

For simplicity putting further $c_2(t) = 0$, we find by the indicated method that

$$f_1 = D(t) - c_1(t) + \frac{B(t) + A(t) - 3D(t) + (3 - M_0^2)\,c_1(t)}{2}\sin^2\theta - N(t)\sin\theta\cos\theta,$$

$$q_1 = \frac{B(t) + A(t) - 3D(t) + (3 - M_0^2)\,c_1(t)}{2}\sin\theta\cos\theta + N(t)\sin^2\theta,$$

$$h_1 = c_1(t)g_0(t)V_0(t)\cos\theta,$$

$$g_1 = \frac{g_0(t)}{V_0(t)}\{M_0^2(t)c_1(t) - B(t)\}\cos\theta - \frac{g_0(t)}{V_0(t)}c_0(t)\sin\theta.$$

Here

$$A(t) = -\frac{1}{g_0(t)}\frac{dg_0(t)}{dt}, \qquad B(t) = \frac{1}{g_0(t)}\frac{dg_0(t)}{dt} - \frac{1}{\gamma h_0(t)}\frac{dh_0(t)}{dt},$$

$$D(t) = -\frac{1}{V_0(t)}\frac{dV_0(t)}{dt};$$

$c_0(t)$, $c_1(t)$, and $N(t)$ are arbitrary functions of time.

As was shown above, the determination of any subsequent coefficients in the expansions (2) is likewise reduced to quadratures.

In conclusion, let us note that if the assumption is not made that the zero terms of the expansions (2) differ from zero, then the following two solutions of system (3) should be considered:

1) $g_0(t) \neq 0, \quad f_0(t,\theta) = q_0(t,\theta) = 0;$

2) $g_0(t) \equiv 0, \quad f_0(t,\theta) = q_0(t,\theta) \equiv 0.$

In these cases expansions of the form (2) also exist, but finding their coefficients is reduced to solving partial differential equations. In the presence of central symmetry, the coefficients of such expansions are found from solutions of systems of ordinary differential equations; moreover, obviously, among such expansions are contained those established earlier by L. I. Sedov (1).

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1. L. I. Sedov, DAN, **85**, No. 4 (1952).

Note: Figure translations are in progress. See original paper for figures.

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