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Abstract

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On Unitary Representations of Nilpotent Lie Groups

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In Dixmier' s paper (^{1}), certain unitary irreducible representations of nilpotent Lie groups, called by the author **special** representations, were investigated.* In particular, it was shown that:

A. The representations are realized in the space of all square-summable functions in Euclidean space R^m, in such a way that the image of the associative envelope of the Lie algebra of the group $\mathfrak{G}$ is the algebra $\mathfrak{D}_m$ of all differential operators with polynomial coefficients.

B. For "sufficiently good" functions φ on $\mathfrak{G}$, the operator

$$T_\varphi = \int_{\mathfrak{G}} \varphi(g) T_g \, dg$$

is completely continuous and has finite trace.

In the present note these results are extended to arbitrary unitary irreducible representations of connected nilpotent Lie groups.**

Let us consider two countably normed spaces: the space E of infinitely differentiable functions on the group $\mathfrak{G}$, with the system of norms

$$\|\varphi\|_q = \max_{\sum k_i \le q} \int_{\mathfrak{G}} \left| X_1^{k_1} \cdots X_n^{k_n} \varphi(g) \right| dg$$

(here $X_1, \dots, X_n$ are the Lie operators on the group $\mathfrak{G}$), and the space $S(R^m)$ of infinitely differentiable functions in the Euclidean space R^m, with the system of norms

$$\|f\|_q^2 = \max_{\sum (k_i + l_i) \le q} \int_{R^m} \left| r_1^{k_1} \cdots r_m^{k_m} \frac{\partial^{l_1 + \cdots + l_m}}{\partial r_1^{l_1} \cdots \partial r_m^{l_m}} f(r_1, \dots, r_m) \right|^2 dr_1 \cdots dr_m .$$

Theorem 1. Statement B follows from statement A, if the functions belonging to E are called "sufficiently good."

Lemma. If an operator A maps the unit sphere of the space $\mathcal{L}^2(R^m)$ into a bounded subset of $S(R^m)$, then for any orthonormal basis $\{f_k\}$ the series

$$\sum_{k=1}^{\infty}(Af_k, f_k)$$

converges absolutely.

Proof. Take an arbitrary differential operator with polynomial coefficients whose inverse is— is a Hilbert-Schmidt operator. (For example, $L = (1 + r^{2n})[E + (-\Delta)^n](1 + r^{2n})$, where $r = (\sum r_i^2)^{1/2}$, Δ is the Laplace operator, and E is the identity operator.) Denote by H_k the Hilbert space obtained by completing $S(R^m)$ with respect to the scalar product

$$(f, g)_k = \int_{R^m} L^k f \cdot \overline{L^k g}\, dr_1 \dots dr_m.$$

The operator A maps H_0 continuously into $S(R^m)$, and therefore it may be regarded as a bounded operator $\widetilde{A}$ from H_0 to H_2. Denote by J_k the imbedding operator of H_k into H_{k-1}. Then the operator A, if regarded as an operator in H_0, decomposes into the product $\widetilde{A} = J_1 J_2 \widetilde{A} = J_1 B$, where J_1 and B are Hilbert-Schmidt operators (J_1 is isometric to L^{-1}, while B is the product of the Hilbert-Schmidt operator J_2 and the bounded operator $\widetilde{A}$). Choose in each H_k an orthonormal basis; for the coefficients of the operator A we obtain the expression

$$A_{mn} = \sum_{p=1}^{\infty}(J_1)_{mp} B_{pn}.$$

Hence

$$\sum_{m=1}^{\infty}|A_{mm}| = \sum_{m=1}^{\infty}\left|\sum_{p=1}^{\infty}(J_1)_{mp}B_{pm}\right| \leq$$

$$\leq \sum_{m,p}\left|(J_1)_{mp}B_{pm}\right| \leq \left(\sum_{m,p}|(J_1)_{mp}|^2\right)^{1/2}\left(\sum_{m,p}|B_{pm}|^2\right)^{1/2} < \infty,$$

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* These representations by no means exhaust all unitary irreducible representations of nilpotent Lie groups. For connected, but not simply connected, groups, special representations may occur in the decomposition of the regular representation.

** *Note added in proof.* This result was obtained by a somewhat different method in Dixmier' s latest paper (^{3}).

as was required to be proved.

To prove Theorem 1 it remains to show that, if φ belongs to E, then the operator T_φ maps the unit sphere of $\mathcal{L}^2(R^m)$ into a bounded subset of $S(R^m)$. This follows from the inequalities

$$\|T_\varphi f\|_q^2 = \max_{k+l\leq q} \int_{R^m} |L_{k,l} T_\varphi f|^2 \, dr = \max_{k+l\leq q} \int_{R^m} |T_{D_{k,l}\varphi} f|^2 \, dr =$$

$$= \max_{k+l\leq q} \|T_{D_{k,l}\varphi}\|^2 \|f\|^2 \leq \|f\|^2 \left(\max_{k+l\leq q} \int_{\mathfrak{G}} |D_{k,l}\varphi| \, dg \right)^2 \leq C\|f\|^2 |\varphi|_p^2$$

(where by $L_{k,l}$ we denote the operator

$$r_1^{k_1} \cdots r_m^{k_m} \frac{\partial^{l_1+\cdots+l_m}}{\partial r_1^{l_1} \cdots \partial r_m^{l_m}};$$

by $D_{k,l}$, an element of the associative envelope which under the representation T_g passes into $L_{k,l}$; and C and p are constants depending on q).

Theorem 2. *Statement A is true for all unitary irreducible representations of connected nilpotent Lie groups.*

Proof. For groups of dimension 1 the theorem is trivial. Suppose it is true for groups of dimension less than n, and consider a group $\mathfrak{G}$ of dimension n. If $\mathfrak{G}$ is not simply connected, extend T_g to a representation $\widetilde{T}_g$ of the simply connected covering group $\widetilde{\mathfrak{G}}$ of the group $\mathfrak{G}$. Since the Lie algebras of the groups $\mathfrak{G}$ and $\widetilde{\mathfrak{G}}$ coincide, the assertion of the theorem for the group $\mathfrak{G}$ follows from the validity of the theorem for simply connected groups.

If the kernel of the representation T_g is not discrete, denote by $\mathfrak{G}'$ the connected component of the identity in the kernel. The assertion of the theorem for the group $\mathfrak{G}$ and the representation T_g follows from the validity of the theorem for the group $\mathfrak{G}/\mathfrak{G}'$, which has smaller dimension. It remains to consider the case when $\mathfrak{G}$ is simply connected and the representation T_g is locally faithful. In this case, as shown in (2), the representation T_g is constructed as follows. Let G be the Lie algebra of the group $\mathfrak{G}$. Then in G one can choose a basis consisting of the vectors $x, y, z, t_1, \dots, t_{n-3}$ with the relations $[x,z] = [y,z] = [t_k, z] = [t_k, y] = 0$,

$[x,y] = z$. Denote by G_0 the subalgebra spanned by the vectors $y, z, t_1, \dots, t_{n-3}$. The representation T_g is induced by some representation U_g of the subgroup $\mathfrak{G}_0 = \exp G_0$ such that $U_{\exp(\tau y)} = E$, $U_{\exp(\tau z)} = e^{i\lambda\tau} E$, $\lambda \neq 0$. By the induction assumption, the representation U_g is realized in the space of square-summable functions on R^m. Then T_g acts in the space of square-summable functions on R^{m+1} by the formula

$$T_{g_0 \exp(\tau x)} f(r_1, r_2, \dots, r_m; r) = U_{\exp(rx) g_0 \exp(-rx)} f(r_1, r_2, \dots, r_m; r + \tau).$$

Hence

$$T_x = \frac{\partial}{\partial r}, \quad T_y = i\lambda r, \quad T_z = i\lambda, \quad T_{t_k} = U_{\exp \operatorname{ad}(rx) t_k} = \sum_j a_{kj} U_{t_j} + c_k,$$

where a_{kj} and c_k are polynomials in r. Denote by $\mathfrak{D}$ the algebra generated by the operators $T_x, T_y, T_z, T_{t_1}, \dots, T_{t_{n-3}}$. Since $r = \frac{1}{i\lambda} T_y$ and $1 = \frac{1}{i\lambda} T_z$ belong to $\mathfrak{D}$, $\mathfrak{D}$ contains all polynomials in r. The matrix $\{a_{kj}(r)\}$ has inverse $\{b_{kj}(r)\} = \{a_{kj}(-r)\}$. Therefore, together with T_{t_j}, the algebra $\mathfrak{D}$ also contains $\sum_j b_{kj} T_{t_j} = U_{t_k} + c_k$, and consequently also U_{t_k}. But $U_y, U_z, U_{t_1}, \dots, U_{t_{n-3}}$ generate $\mathfrak{D}_m$. Thus $\mathfrak{D} \supset \mathfrak{D}_m$. Since, moreover, $\frac{\partial}{\partial r} = T_x \in \mathfrak{D}$, it follows that $\mathfrak{D} = \mathfrak{D}_{m+1}$, as was required to prove.

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[2] A. A. Kirillov, DAN, 128, No. 5 (1959).
[3] J. Dixmier, Bull. Soc. Math. France, 87, No. 1 (1959).

Note: Figure translations are in progress. See original paper for figures.

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