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Abstract

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF WEAKLY NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Let $\mathfrak{X}$ be an n-dimensional vector space endowed with an arbitrary norm; let T be a finite or infinite interval of the real axis; $x \in \mathfrak{X},\ t \in T$. Consider the system of differential equations

$$\frac{dx}{dt} = A(x,t)x + f(x,t), \tag{1}$$

where A is a matrix of order n. If A is "almost constant," and f is small, then it is natural to compare the solution of system (1) with the solutions of the systems

$$\frac{dx}{dt} = A(x_0,t_0)x \tag{2}$$

(the "freezing" method). As is known, the solutions of systems of the form (2) are determined by the formula $x(t) = \exp\{A(x_0,t_0)(t-t_0)\}x(t_0)$, and their behavior depends on the characteristic roots $\lambda_i(A)$ of the matrix A. In what follows it is assumed that the norm of matrices is induced in the usual way by the norm of vectors in $\mathfrak{X}$. By the symbols $\|x\|_D$ and $\|A\|_D$ we denote norms defined with the aid of a positive-definite matrix D: $\|x\|_D^2 = (Dx,x)$.

In addition, introduce the notation $\Lambda(A) = \max \operatorname{Re} \lambda_i(A)$. The following inequalities hold:

$$\|e^{At}\| \le e^{\|A\|t}; \tag{3}$$

$$\|e^{At}\| \le \sum_{k=0}^{n-1} \frac{(2t\|A\|)^k}{k!}\, e^{\Lambda(A)t}; \tag{4}$$

$$\|e^{At}\|_D \leq e^{\frac{1}{2}\Lambda(DA+A^*D)t}. \tag{5}$$

The first of these is well known. The proof of inequality (4) is given in ([4]), p. 78; however, in the estimate presented in that book the factors $1/k!$ are absent. By a more careful estimate of the coefficients denoted in ([4]) by b_k, one can verify the validity of inequality (4). Inequality (5) is well known in stability theory (the quadratic form (Dx, x) plays the role of a Lyapunov function for the case when the first-approximation system is linear).

The purpose of the present note is to give at least a rough estimate of the influence of the dependence of A on x and t on the growth of solutions. Theorem 1 makes it possible to reduce the derivation of such an estimate to a Volterra integral equation, which in the one-dimensional-in-time case is conveniently solved with the aid of the Laplace transform. Theorem 2 concerns the case when A does not depend on x, and estimates the growth exponent of solutions of system (1) in terms of the maximal growth exponent of the solution of the "frozen" systems. The estimates obtained are more general and

more precise analog of (2). Theorem 3 pertains to the general case. The dependence of A on x causes a peculiar interlacing of the conditions, expressed by condition 5) of this theorem. Theorem 4, which slightly generalizes one of the theorems of N. N. Krasovskii (3), gives a concrete example of the application of the preceding theorem. Namely, if, for some choice of the norm $\|x\|_D$, the frozen systems realize a "contracting" mapping (i.e., if the norm of all their solutions strictly decreases), then for small f the same is also true for the solutions of system (1). If $\Lambda(A) < 0$ and A is constant, then such a norm can always be chosen. However, in the case of variable $A(x,t)$ this is a new requirement, which must be imposed on A together with the requirement of smooth variation. At the same time, however, one can do without the requirement that the Lipschitz constant be small.

In conclusion, we note that Theorems 1—3 carry over without changes to Banach spaces.

2. Let $x(s)$ be an absolutely continuous vector, and let $B(s)$ be an absolutely continuous matrix. Denote

$$y(t,s) = \exp\{B(s)(t-s)\}x(s).$$

It is easily verified that the following identity holds almost everywhere:

$$\frac{\partial y(t,s)}{\partial s} = \int_s^t e^{B(s)(t-\sigma)} \frac{dB(s)}{ds} e^{B(s)(\sigma-s)}\, d\sigma \cdot x(s) + e^{B(s)(t-s)} \left[\frac{dx(s)}{ds} - B(s)x(s)\right]. \tag{6}$$

Moreover, in many cases the following obvious lemma is useful:

Lemma 1. If L is a linear operator in a space of functions such that $R = (E-L)^{-1} > 0$ (i.e., from $f > 0$ it follows that $Rf > 0$), $f = \varphi + Lf$, $g \geq \varphi + Lg$, then $g \geq f$.

In particular, the lemma is true if $L > 0$ and the series $\sum_{k=0}^{\infty} L^k$ converges*.

From identity (6) and Lemma 1 there follows the following theorem:

Theorem 1. Suppose that on the interval $[t_0, t_1] \subset T$ $x(t)$ and $B(t) = A(x(t), t)$ are absolutely continuous, $t_0 \leq s \leq t_1$, $0 \leq \tau \leq t_1 - s$, and moreover:

1)
$$\left\| \frac{dB(s)}{ds} \right\| \leq \psi(s);$$

2)
$$\left\| \frac{dx(s)}{ds} - B(s)x(s) \right\| \leq \varphi(s);$$

3)
$$\| \exp\{B(s)\tau\} \| \leq \eta(\tau, s);$$

4) the functions $\varphi(s)$ and $\psi(s)$ are integrable on $[t_0, t_1]$, and $\eta(\tau, s)$ is bounded;

5)
$$K(t,s) = \int_s^t \eta(t-\sigma, s)\eta(\sigma - s, s)\, d\sigma \quad \text{for} \quad t_0 \leq s \leq t \leq t_1.$$

Then for all $t \in [t_0; t_1]$ the inequality

$$\|x(t)\| \leq g(t),$$

holds, where $g(t)$ is the solution of the integral equation

$$g(t) = \eta(t-t_0, t_0)\|x(t_0)\| + \int_{t_0}^t \varphi(s)\eta(t-s,s)\, ds + \int_{t_0}^t K(t,s)\psi(s)g(s)\, ds. \quad (7)$$

* If, for example, one puts $Lf = \int_0^t v(s)f(s)\, ds$, where $v(s) \geq 0$, then from Lemma 1 there immediately follows Bellman' s "basic lemma" ((1), p. 46).

For the proof it is enough to note that

$$\mathbf{x}(t) = \exp\{\mathbf{B}(t_0)(t-t_0)\}\mathbf{x}(t_0) + \int_{t_0}^t \frac{\partial \mathbf{y}(t,s)}{\partial s}\, ds$$

and, estimating the right-hand side by means of (6), to use Lemma 1.

Equation (7) has an especially simple form in the case when $\eta(t, s)$ does not depend on s and $\mathbf{B}(t)$ satisfies the Lipschitz condition with constant δ. Denote $\varphi(t_0 + \tau) = \varphi_1(\tau)$, $g(t_0 + \tau) = g_1(\tau)$. Equation (7) now takes the form

$$g_1 = \eta\|\mathbf{x}(t_0)\| + \varphi_1 * \eta + \delta(\eta * \eta * g_1),$$

where $*$ is the convolution symbol. If $\Phi(p)$, $H(p)$, and $G(p)$ are the Laplace transforms of the functions φ_1, η, and g_1, respectively, then

$$G = H \cdot \|\mathbf{x}(t_0)\| + H \cdot \Phi + \delta H^2 \cdot G, \tag{8}$$

and, consequently, G can be found explicitly, while g_1 is computed by the formulas for inversion of Laplace transforms.

3. We proceed to concrete examples of the application of the preceding estimate. Let the system under consideration be

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(\mathbf{x}, t). \tag{9}$$

Theorem 2. If for all $t, t' \geq t_0$, $s \geq 0$:

1) $\|\exp\{\mathbf{A}(t)s\}\| \leq \eta(s)$;
2) $\|\mathbf{f}(\mathbf{x}, t)\| \leq \omega\|\mathbf{x}\|$;
3) $\|\mathbf{A}(t') - \mathbf{A}(t)\| \leq \delta|t' - t|$

and λ is such that the inequality

$$\int_0^\infty e^{-\lambda s}\eta(s)\, ds \leq \frac{2}{\omega + \sqrt{\omega^2 + 4\delta}}, \tag{10}$$

is satisfied, then there exists a $K > 0$ such that all solutions of system (9) satisfy the inequality

$$\|\mathbf{x}(t)\| \leq K e^{\lambda(t-t_0)}\|\mathbf{x}(t_0)\|. \tag{11}$$

If $\eta(s)$ is taken in explicit form (using, for example, inequalities (3)–(5)), then various estimates can be obtained for λ. Let, in particular, as in (2), $\|\mathbf{A}(t)\| \leq M$, $\mathbf{f} = 0$, and $\Lambda(\mathbf{A}(t)) \leq -\gamma$. Then, according to (4),

$$\eta(s) = \sum_{k=0}^{n-1} \frac{(2Ms)^k}{k!} e^{-\gamma s},$$

and inequality (10) leads us to the condition

$$(\lambda+\gamma)^n \geq \sqrt{\delta}\sum_{k=0}^{n-1}(\lambda+\gamma)^{n-k-1}(2M)^k.$$

Theorem 2 from (2) ensures the fulfillment of (11) for $\lambda = -\delta/4$. For such a λ our theorem is applicable if, for example,

$$\delta < \left(\frac{3}{8}\right)^{2n}\frac{\gamma^{2n}}{M^{2n-2}},$$

whereas the theorem of N. Ya. Lyashchenko mentioned above requires the fulfillment of the significantly more restrictive condition, for all n,

$$\delta < \frac{\gamma^{2n+2}}{4^{2n+1}n^2M^{2n}(n-1)^{n-1}\ln\left[M^n 2^{2n} n\sqrt{(n-1)^{n-1}\gamma^{-n}}\right]}.$$

4. In the case when the matrix $\mathbf{A}$ depends explicitly on $\mathbf{x}$, additional difficulties arise, connected with the estimate of the matrix

$$\mathbf{C}(\mathbf{x},s) = \frac{d}{ds}\mathbf{A}(\mathbf{x}(s),s) = (C_{ij}),$$

$$C_{ij}(\mathbf{x},s) = \frac{\partial a_{ij}}{\partial t} + \sum_{k,l}\frac{\partial a_{ij}}{\partial x_k}a_{kl}x_l.$$

Condition 3) of Theorem 2 can be satisfied here only if the partial derivatives $\partial a_{ij}/\partial t$, $\partial a_{ij}/\partial x_k$ tend to zero sufficiently rapidly as $\|x\| \to \infty$. If this is so, then Theorem 2 remains valid in this case as well. In the general case, new conditions are needed.

Theorem 3. Let, for some positive L, q, τ, in the region of variation of the variables x, t defined by the inequality $\|x\| < Lq^{t-t_0}$, the following conditions be satisfied for all $s \geq 0$:

1) $\|C(x,s)\| \leq \delta$;
2) $\|f(x,t)\| \leq \omega\|x\|$;
3) $\exp\{A(x,t)s\} \leq \eta(s)$, and the function $\eta(s)$ has as its Laplace transform $H(p)$;
4) $g_1(t)$ has as its Laplace transform the function

$$G(p) = H(p)/[1-\omega H(p) - \delta H^2(p)];$$

5) $g_1(\tau) < q^\tau$;

6) $\sup_{0 \le t \le \tau} g_1(t)/g_1(\tau)^{(t-t_0)/\tau} = l$.

Then all solutions of system (1) such that $\|x(t_0)\| \le L/l$ satisfy, for $t \ge t_0$, the inequality

$$\|x(t)\| \le l g_1(\tau)^{(t-t_0)/\tau} \|x(t_0)\|. \tag{12}$$

From Theorem 3 there easily follows an assertion generalizing, in a certain sense, the theorem of N. N. Krasovskii ((3), p. 109, Theorem 21.1).

Theorem 4. Let the norm in $\mathfrak{X}$ be given by means of a positive-definite quadratic form (Dx, x), and let $f_1(x,t)$ be such that there exists a matrix $A(x,t)$ satisfying, in the region $t \ge t_0$, $\|x\| < L$, the conditions:

1) $\Lambda(DA + A^* D) \le -2\gamma < 0$;

2) $\|f_1(x,t) - A(x,t)x\|_D \le \omega \|x\|_D < \gamma \|x\|_D$;

3) $\|A(x', t') - A(x'', t'')\|_D \le K(\|x' - x''\|_D + |t' - t''|)$;

4) $\|A(x,t)\|_D \le M$.

Then all solutions of the system $dx/dt = f_1(x,t)$ such that $\|x(t_0)\|_D \le L$ tend to zero.

We note that we impose no conditions on $f_1(x,t)$ ensuring uniqueness of solutions or their existence for all initial conditions.

Remark. In the cited theorem of N. N. Krasovskii, $f_1(0) = 0$, and condition 1) is satisfied by the matrix $\partial f/\partial x$. Put

$$A_0(x,t) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\, d\sigma.$$

Then $f_1(x,t) = A_0(x,t)x$. Taking into account that, for symmetric matrices S,

$$\Lambda(S) = \sup \frac{(Sx,x)}{(x,x)},$$

we conclude that A_0 also satisfies condition 1). Approximating A_0 now by a matrix A satisfying the Lipschitz condition, we obtain the fulfillment of conditions 1)–3) with a somewhat smaller γ. Thus, the conditions of Theorem 4 are broader than the conditions of Krasovskii' s theorem.

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Note: Figure translations are in progress. See original paper for figures.