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Abstract

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THE DIRICHLET PROBLEM FOR A CLASS OF ELLIPTIC SYSTEMS

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It is known that the Dirichlet problem (the first boundary value problem) for linear systems of elliptic type in the sense of I. G. Petrovskii is, generally speaking, not a problem of Fredholm type ([1]). The Fredholm alternatives for the indicated problem always hold if the usual ellipticity requirement is strengthened by the condition of strong ellipticity ([2]). It is of interest to study the Dirichlet problem for elliptic systems without imposing the condition of strong ellipticity.

Let there be given a system of linear partial differential equations of the second order of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0, \tag{1}$$

where $A = \|A_{ik}\|$, $B = \|B_{ik}\|$, $C = \|C_{ik}\|$ are given real constant square matrices of order n; $u = (u_1, u_2, \dots, u_n)$ is the vector sought.

The system (1) is called elliptic in the sense of I. G. Petrovskii if all $2n$ roots of the characteristic equation

$$\det|A + 2B\lambda + C\lambda^2| = 0 \tag{2}$$

are complex (non-real).

Denote by $\lambda_1, \dots, \lambda_\mu$, $\bar{\lambda}_1, \dots, \bar{\lambda}_\mu$ the roots of equation (2), and by $k_1, \dots, k_\mu$ the multiplicities of these roots.

The general representation of regular solutions of system (1) in a finite simply connected domain D has the form ([3])

$$u = \operatorname{Re} \sum_{j=1}^{\mu} \sum_{l=1}^{k_j} \sum_{m=0}^{l-1} C_{lm}^{(j)} \bar{z}_j^m \varphi_{jl}^{(m)}(z_j), \tag{3}$$

where the φ_{jl} are arbitrary holomorphic functions in the domain D of the variables $z_j = x + \lambda_j y$, respectively; the upper index m of the function φ_{jl} indicates the order of derivative with respect to z_j, and $C_{lm}^{(j)}$ are completely determined constant vectors, expressible solely in terms of the coefficients of system (1), with the determination of $C_{lm}^{(j)}$ requiring the solution of linear algebraic systems whose coefficients are expressed through the coefficients of system (1).

The Dirichlet problem for system (1) consists in determining a solution of this system, regular in the domain D, continuous up to the contour Γ of the domain D, and satisfying the condition

$$u = f \quad \text{on } \Gamma, \tag{4}$$

where $f = (f_1, \dots, f_n)$ is a given vector.

In the present note we confine ourselves to the study of the Dirichlet problem for system (1) in the case $n = 2$ in a circular domain, when equation (2) has a multiple root.

Let λ be a double root of equation (2). Without loss of generality one may assume that $\lambda = i$. This can always be achieved by a nonsingular affine transformation of the independent variables; however, if before the change of variables the domain D was the disk $|z| < 1$, then after the transformation it will, generally speaking, become an ellipse.

Thus, we shall assume that $\lambda = i$ is a double root of equation (2), and that the domain D coincides with the disk $|z| < 1$. Suppose also that the vector $f = (f_1, f_2)$ prescribed on the circumference Γ of the disk D satisfies the Hölder condition.

In the case under consideration, the general representation of the solutions of system (1) regular in the domain D, in view of (3), has the form

$$u = \operatorname{Re}\left[a\Phi(z) + a_1\Psi(z) + a_2\bar{z}\Psi'(z)\right], \tag{5}$$

where $\Phi(z)$ and $\Psi(z)$ are functions of the variable $z = x + iy$, holomorphic in D, and the vectors a, a_1, a_2 are solutions of the system

$$\begin{gathered}(A + 2Bi - C)a = 0, \qquad (A + 2Bi - C)a_2 = 0, \\ (A + 2Bi - C)a_1 + 2(A + C)a_2 = 0.\end{gathered} \tag{6}$$

The rank r of the matrix $A + 2Bi - C$ is either 1 or 0. In the case $r = 1$, choosing a definite (nonzero) value for the vector a, we shall have $a_2 = \mu a$, where μ is a scalar. For the third group of equations (6), with respect to the vector a_1, the solvability conditions will be satisfied, and we can write $a_1 = \nu a + \mu b$, where

μb is a particular solution of the nonhomogeneous system of this same group of equations.

Consequently, the general representation (5) can be rewritten in the form

$$u = \operatorname{Re}\,[a\varphi(z) + b\psi(z) + a\bar{z}\psi'(z)], \tag{7}$$

where $\Phi(z) + \nu\Psi(z) \equiv \varphi(z)$, $\mu\Psi(z) \equiv \psi(z)$.

If $r = 0$, then obviously $B = 0$, $A = C$, and consequently system (1) has the form $A(u_{xx} + u_{yy}) = 0$. But $\det A \neq 0$. Hence we conclude that system (1) is split: $\Delta u_i = 0$ $(i = 1, 2)$, where Δ is the Laplace operator. Therefore we immediately conclude that for $r = 0$ the Dirichlet problem for system (1) is always solvable, and in a unique way.

It remains to consider the case $r = 1$. We shall start from the general representation (7). Equating the expression for $u(x, y)$ from (7) on the circumference Γ to the function f, we readily obtain the relation

$$a\varphi(z) + b\psi(z) + a\frac{\psi'(z) - \psi'(0)}{z} + \overline{a\varphi(0)} + \overline{b\psi(0)} + \overline{a\psi''(0)} + \overline{a\psi'(0)}z = \\ = \frac{1}{\pi i}\int \frac{f(t)\,dt}{t - z}. \tag{8}$$

Following (3), we shall call system (1) **weakly coupled** if the vectors a and b are linearly independent. Otherwise system (1) will be called **strongly coupled**. In view of the fact that

$$a\varphi(0) + b\psi(0) + a\psi''(0) + \overline{a\varphi(0)} + \overline{b\psi(0)} + \overline{a\psi''(0)} = \frac{1}{\pi i}\int \frac{f\,dt}{t}, \\ a\varphi'(0) + b\psi'(0) + \frac{a}{2}\psi'''(0) + \overline{a\psi'(0)} = \frac{1}{\pi i}\int \frac{f\,dt}{t^2}, \tag{9}$$

relation (8) can be rewritten in the form

$$a[\varphi(z) - \varphi(0) - \varphi'(0)z] + b[\psi(z) - \psi(0) - \psi'(0)z] + \\ + a\frac{\psi'(z) - \psi'(0) - \psi''(0)z - \frac{1}{2}\psi'''(0)z^2}{z} = \frac{1}{\pi i}\int_\Gamma \left(\frac{1}{t - z} - \frac{1}{t} - \frac{z}{t^2}\right) f(t)\,dt. \tag{10}$$

In the case of a weakly coupled system (1), for the homogeneous Dirichlet problem ($f = 0$), from (10) we obtain

$$\psi(z) = \psi(0) + \psi'(0)z, \qquad \varphi(z) = \varphi(0) + \varphi'(0)z.$$

Hence it follows that the homogeneous Dirichlet problem under consideration has only the trivial solution

$$u_0(x,y) = \operatorname{Re}\{a[\varphi(0) + \varphi'(0)z] + b[\psi(0) + \psi'(0)z] + \bar{a}z\psi'(0)\} = 0.$$

Using formula (10) again, we conclude that the inhomogeneous Dirichlet problem always has a unique solution, and it is written explicitly in the form

$$u(x,y) = (z\bar{z} - 1)\operatorname{Re}\left\{aM\frac{1}{\pi i}\int_\Gamma \frac{f_1(t)(2t-z)}{t^2(t-z)^2}\,dt + \frac{aN}{\pi i}\int_\Gamma \frac{f_2(t)(2t-z)}{t^2(t-z)^2}\,dt\right\} + \operatorname{Re}\frac{1}{\pi i}\int_\Gamma \frac{f\,dt}{t-z} - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\,dt}{t}, \tag{11}$$

where

$$M = -\frac{a_2}{a_1b_2 - a_2b_1}, \qquad N = \frac{a_1}{a_1b_2 - a_2b_1}.$$

If system (1) is strongly coupled, we have $b = \mu a$, where μ is a scalar parameter. In this case, by virtue of (10), the inhomogeneous Dirichlet problem will be solvable only under the identical fulfillment of the condition

$$\int_\Gamma \left(\frac{1}{t-z} - \frac{1}{t} - \frac{z}{t^2}\right)(a_2f_1(t) - a_1f_2(t))\,dt = 0. \tag{12}$$

Identity (12) is obviously equivalent to the conditions

$$\int_\Gamma (\bar{a}_2f_1(t) - \bar{a}_1f_2(t))t^k\,dt = 0, \qquad k = 1, 2, \dots, \tag{13}$$

and, if conditions (13) are satisfied, then the solution of the inhomogeneous Dirichlet problem exists and is written explicitly as

$$u(x,y) = (z\bar{z} - 1)\operatorname{Re} a\frac{\psi'(z) - \psi'(0)}{z} + \operatorname{Re}\frac{1}{\pi i}\int_\Gamma \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\,dt}{t}.$$

Here the expression

$$u_0(x,y) = (z\bar{z} - 1)\operatorname{Re} a\frac{\psi'(z) - \psi'(0)}{z}$$

for an arbitrary holomorphic function $\psi(z)$ is a solution of the homogeneous Dirichlet problem.

It is not difficult to show that, for weak coupling of system (1), in the case when $\lambda = i$ is a double root and $n = 2$, the necessary and sufficient condition is

$$\det|A + C| \neq 0. \tag{14}$$

In the case when, instead of i, the double root is λ, condition (14) takes the form

$$\det|A + B\lambda + \bar{B}\bar{\lambda} + C\lambda\bar{\lambda}| \neq 0.$$

We have shown that in any finite simply connected domain the weak coupling of system (1) is a condition necessary and sufficient for the Fredholm property of the Dirichlet problem.

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Note: Figure translations are in progress. See original paper for figures.