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Abstract

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GEODESIC FLOWS ON MANIFOLDS OF NEGATIVE CONSTANT CURVATURE

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The construction of a manifold of negative constant curvature as the fundamental domain R of a certain discrete subgroup of the group of motions of the Lobachevsky plane, and of the geodesic flow on it, is described in detail in $(1, 2)$. The method proposed below makes it possible, for a broad class of such flows, to describe completely the spectral type of the corresponding dynamical system and to establish the presence in it of mixing of all degrees (5). The basis of this method is the notion of a Kolmogorov dynamical system (see § 1).

§ 1. Kolmogorov dynamical systems and their properties

Definition. A measurable flow $\{S^t\}$, acting in a Lebesgue space M with measure μ, is called a **Kolmogorov flow** if there exists a measurable partition ζ^0 of the space M satisfying the following three conditions:

1. $\zeta^t = S^t\zeta^0 > \zeta^{t_1}$ for $t > t_1$*.
2. $$\prod_{t=-\infty}^{\infty} \zeta^t = \varepsilon$$ (ε is the partition mod 0 into individual points).
3. $$\bigcap_{t=-\infty}^{\infty} \zeta^t = \nu$$ (ν is the σ-algebra containing the subsets having measure 0 or 1).

Many of the known properties of Kolmogorov flows are collected in (10). Here we shall only list them.

I. The group of unitary operators $\{U_t\}$ corresponding to a Kolmogorov flow has, in the orthogonal complement to the subspace of constants, countably multiple Lebesgue spectrum.

II. Kolmogorov flows have positive entropy.

III. Kolmogorov flows possess mixing of all degrees.

§ 2. The two-dimensional case

Let a surface of negative constant curvature be realized as the fundamental domain R of a certain discrete subgroup Γ of the group of all fractional-linear transformations of the complex plane $z = x_1 + ix_2$ which carry the unit disk onto itself. We shall be interested only in the case when R has finite area. Then, as follows from Siegel' s theorem (11), there exists a geodesically convex fundamental domain in the form of a polygon with a finite number of sides, some of which may, however, have infinite length**. By a choice of the curvature one can always ensure that the volume of the space of line elements is equal to one.

Theorem 1. *Geodesic flows on two-dimensional manifolds of negative constant curvature and finite area are Kolmogorov flows.*

* On the subject of inequalities between partitions see (12).
** Areas, lengths, and other metric quantities refer to the non-Euclidean metric.

The proof of this theorem breaks up into a number of lemmas. Below, M denotes the space of line elements of the surface R.

Lemma 1. There exists a measurable partition ζ of the space M such that:

a) almost every element C of the partition ζ is specified by a quadrilateral whose two lateral sides are segments of parallel, equally directed straight lines, while the other pair of sides is formed by segments of horocycles parallel and orthogonal to them, and consists of line elements whose carriers lie inside this quadrilateral and whose directions are parallel to the lateral sides (i.e., the directed geodesic determined by the given line element ends at the same infinitely distant point as the lateral sides);

b) the maximal length of a side of such a quadrilateral does not exceed, almost everywhere, a prescribed constant A;

c) there exists a constant $B > 0$ such that the measure of the set G_x of those elements of the partition ζ for which the maximal length of a side does not exceed x satisfies the inequality

$$\mu(G_x) < Bx^{1/4};$$

d) there exists a measurable partition $\tilde{\zeta}$ such that almost all its elements are formed by line elements of the surface R whose carriers lie on arcs of horocycles of finite length, and whose directions are orthogonal to such arcs (and are directed toward the infinitely distant point of the horocycle); for some $\Delta > 0$

$$\tilde{\zeta} \geq \prod_{-\Delta<\tau\leq 0} S^{\tau}\zeta;$$

the measure of the set $\widetilde{G}_x$ of those elements of the partition $\tilde{\zeta}$ to which there correspond arcs of a horocycle of length less than x satisfies the inequality

$$\mu\{G_x\} < C_1 x^{1/4},$$

where C_x is some positive constant.

The partition ζ mentioned in Lemma 1 may be constructed in many ways. One of them, briefly, is as follows. Take some point θ on the unit circle and draw through it parallel horocycles passing through the vertices of the fundamental domain or tangent to its sides. Then between the first and the last horocycles there will lie quadrilaterals in which one pair of sides consists of arcs of parallel horocycles. By adding, if necessary, a further finite number of horocycles, one may arrange that in all such quadrilaterals quadrilaterals of the form required by condition a) of the lemma can be inscribed. There will remain triangles in which one side coincides with the boundary of the fundamental domain. Carry out this construction for all values of θ. Then, adjoining to each finite side of the domain R a congruent domain, we obtain, for fixed θ, the same triangles as above, but attached to R on the other side. The construction, from them, of quadrilaterals of the required form is carried out by extending the arcs of horocycles on each side until they meet the side of the triangle on the other side. The partition of the remaining unbounded part of R, and the subsequent formation of the required quadrilaterals, proceeds analogously with the aid of a countable number of equidistant parallel horocycles.

As a result we obtain a partition of the space M satisfying a). It is clear that one can always secure condition b). Condition c) requires that there be not too many quadrilaterals with small sides. The latter occur only inside the angles ending in parabolic vertices, along the boundary of R, and for exceptional values of θ, of sufficiently small measure. An estimate of their total number is obtained by means of simple geometric considerations.

Consider the partition ξ', which is obtained from ξ if each quadrilateral defining an element of the partition ξ is divided by means of horocycle arcs parallel to the base. Put, for $A < \Delta/2$,

$$\tilde{\xi} = \prod_{-\Delta<\tau<0} S^{\tau}\xi'.$$

By the same considerations that were used in verifying condition b), condition c) is verified.

Lemma 2. *Let the partition ξ satisfy conditions a)–c) of Lemma 1.*

Put $\xi^0 = \prod_{\tau<0} S^\tau \xi$. *Then almost every element of the partition* ξ^0 *is formed by linear elements of the surface* R*, whose carriers lie on arcs of horocycles of length not exceeding* A*, and whose directions are orthogonal to such an arc. The set of elements of the partition* ξ^0 *consisting of a single point has measure zero.*

Let us note the main points of the proof of Lemma 2. The fact that two linear elements belonging to one element C of the partition ξ, and having carriers lying on arcs of different horocycles parallel to the base, for almost all C lie in different elements of the partition ξ^0, is obvious. Further, the quadrilateral corresponding to the element C of the partition ξ is transformed under the action of the transformation S^t, $t < 0$, into a quadrilateral of the same form, but stretched in one direction e^t times.* By virtue of properties b) and c) of the partition ξ, the set H_n of those elements of the partition ξ' which are divided during the time from $-n\Delta$ to $-(n+1)\Delta$ satisfies the inequality

$$\mu(H_n) < De^{-n/10},$$

where D is some positive constant. On the basis of the Borel–Cantelli lemma, almost every element of the partition ξ' belongs to only a finite number of the sets H_n.

Lemma 3. *If the partition* ξ^0*, whose elements are formed from linear elements of the surface* R *with carriers on an arc of a horocycle of length not exceeding some constant* A*, and with directions orthogonal to it, satisfies the condition*

$$\xi^t = S^t\xi^0 \geq \xi^0 \quad \text{for } t > 0,$$

then it is a Kolmogorov partition.

(A Kolmogorov partition is a partition satisfying conditions 1–3 of the definition in § 1.)

The proof evidently requires only verification of condition 3, which we formulate in the following form: for any set N of positive measure,

$$\lim_{t\to-\infty} \mu_{S^tC}(N) = \mu(N) **$$

almost everywhere. In this form condition 3 is a simple consequence of the ergodicity of the horocycle flow, established by Hedlund and Hopf (1).

Theorem 1 follows directly from Lemmas 1, 2, 3, since the partition ξ^0 of Lemma 2 satisfies the conditions of Lemma 3.

§ 3. **The n-dimensional case.** The passage to manifolds of higher dimension causes no difficulties if, following Hopf, one extends the geodesic flow in the space of linear elements to the geodesic flow in the space of n-frames. We restrict ourselves to the case when the fundamental domain R of the corresponding subgroup Γ is bounded by a finite number of $(n-1)$-dimensional faces. Passing through R all possible two-dimensional planes determined by n-frames, we shall be able in each of them to cut

* We assume, for simplicity, that the motion takes place with unit speed. In what follows (§ 4) we shall abandon this assumption.

** By $\mu_C(N)$ (respectively $\mu_{S^tC}(N)$) is denoted the conditional measure of the set N under the condition C (respectively S^tC).

constructions analogous to § 2. Since the theorem on the ergodicity of the horocycle flow remains valid, we obtain the following theorem:

Theorem 2. *The geodesic flows in the space of n-frames of n-dimensional manifolds of negative constant curvature* and finite area, representable as a fundamental domain with a finite number of faces of some subgroup Γ, are Kolmogorov flows.*

§ 4. Entropy of geodesic flows.

Consider, on a compact n-dimensional manifold of constant negative curvature $-k$ $(k > 0)$, having volume V, the motion of linear elements along geodesics with velocity w. Using the definitions and results of the papers ($^{6-9}$), we compute the entropy of the corresponding flow $\{S^t\}$.

Theorem 3. *The entropy of the flow $\{S^t\}$ is equal to*

$$h(\{S^t\}) = h(S^1) = \frac{w\sqrt{k}}{\sqrt[n]{V\omega_{n-1}}}\log e,$$

where ω_{n-1} is the area of the $(n-1)$-dimensional unit sphere and log *is taken to base* 2.

Proof. Let θ be a point on the unit sphere. For all θ, partition the region R into pieces of spherical surfaces of n-dimensional spheres internally tangent to the unit sphere at the point θ. The partition into linear elements with carriers on the resulting pieces of spheres and with directions orthogonal to such spheres will be denoted by ζ. Denote by ζ_ε the partition obtained if each piece of a spherical surface is divided into parts of area equal to ε (the parts on the boundary may also have smaller area). It is easy to show that, for sufficiently small Δ and as $\varepsilon \to 0$, the expression

$$\frac{1}{\Delta}h(S^\Delta\zeta_\varepsilon \mid \zeta_\varepsilon \vee S^{-\Delta}\zeta_\varepsilon \vee \cdots \vee S^{-n\Delta}\zeta_\varepsilon \vee \cdots)$$

tends to

$$\frac{w\sqrt{k}}{\sqrt[n]{V\omega_{n-1}}}\log e.$$

On the other hand, for small ε the partitions ζ_ε have the following property: there exists a sequence of finite partitions $\xi_\varepsilon^1 \leqslant \xi_\varepsilon^2 \leqslant \cdots$, $\prod_n \xi_\varepsilon^n = \zeta_\varepsilon$, such that

$$\lim_{n\to\infty} h(\xi_\varepsilon^n, S^\Delta) = \lim_{n\to\infty} H(S^\Delta \xi_\varepsilon^n \mid \xi_\varepsilon^n \vee S^{-\Delta}\xi_\varepsilon^n \vee \cdots \vee S^{-i\Delta}\xi_\varepsilon^n \vee \cdots) =$$

$$= H(S^\Delta \zeta_\varepsilon \mid \zeta_\varepsilon \vee S^{-\Delta}\zeta_\varepsilon \vee \cdots \vee S^{-i\Delta}\zeta_\varepsilon \vee \cdots).$$

Theorem 1 of ([9]) asserts that in this case the entropy of the automorphism S^Δ can be computed along the sequence of partitions ξ_ε^n as $n \to \infty$ and $\varepsilon \to 0$, or, by virtue of the last equality, along the partitions ζ_ε.

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[*] Constancy of curvature in the n-dimensional case is understood as constancy of the curvature of all possible two-dimensional geodesic surfaces on the given n-dimensional manifold.

Note: Figure translations are in progress. See original paper for figures.

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