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V. M. VOLOSOV

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Abstract

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V. M. VOLOSOV

AVERAGING OF CERTAIN PERTURBED MOTIONS

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§ 1. **Statement of the problem.** Consider an unperturbed system of the form

$$\dot{x} = F(x, y), \qquad \dot{y} = \Phi(x, y), \tag{1}$$

where x, F and y, Φ are, respectively, n-dimensional and m-dimensional vector functions. Let F, Φ be periodic in $y_1, y_2, \dots, y_m$, respectively with constant periods $T_1, T_2, \dots, T_m \neq 0$. Suppose that, in some domain, system (1) has a general solution depending on $n+m$ arbitrary constants and possessing the following property: x_i $(i = 1, 2, \dots, n)$ are periodic in t with period T_0 $(0 \leq T_0 < \infty)$, depending on $n+m-1$ arbitrary constants, while y_j $(j = 1, 2, \dots, m)$ acquire over the time $\Delta t = T_0$ increments equal, respectively, to $T_1, T_2, \dots, T_m$—the periods of F, Φ in the variables $y_1, y_2, \dots, y_m$. We shall call the coordinates x_i **oscillatory**, and the y_i **rotational**.

Under the assumptions made, the solution of (1) is expressed in the form

$$x = x_0(c, \psi), \qquad y = \frac{T\psi}{2\pi} + y_0(c, \psi), \tag{2}$$

where $c = \{c_1, c_2, \dots, c_{n+m-1}\}$ is the collection of $n+m-1$ constants on which the period T_0 depends; $\psi = \omega(c)t + h$ is the phase of the oscillations; h is the n-th arbitrary constant; $\omega(c) = 2\pi/T_0(c)$ is the angular frequency; $T_0(c)$ is the period of the oscillations; $T = \{T_1, T_2, \dots, T_m\}$ is the collection of constant periods of F, Φ in the variables y_j, and the vector functions x_0, y_0 are periodic in ψ with period 2π. Suppose that in the domain under consideration the general solution (2) of system (1) corresponds to a complete system of independent integrals

$$c = P(x, y), \qquad \psi = \theta(x, y), \tag{3}$$

where $P = \{P_1, P_2, \dots, P_{n+m-1}\}$, and P, θ are periodic in y_j with the same constant periods as the functions F, Φ.

Consider a perturbed system of the form

$$\dot{x} = F(x, y) + \varepsilon f(x, y, \varepsilon), \qquad \dot{y} = \Phi(x, y) + \varepsilon \varphi(x, y, \varepsilon), \tag{4}$$

where $\varepsilon \neq 0$ is a small parameter, and f and φ are periodic in y_j with periods T_j. The solutions of (4) will be considered in a neighborhood of solutions (2) of system (1), for small ε, on large time intervals $t \sim 1/\varepsilon$. By variation of the constants $c_1, c_2, \dots, c_{n+m-1}$ and of the phase ψ, we shall seek a representation of the solutions of (4) in the form (2), assuming that $c = c(t, \varepsilon)$, $\psi = \psi(t, \varepsilon)$ are new unknown functions to be determined.

Since the integrals (3) correspond to the solutions (2) of system (1), the unknown functions $c(t, \varepsilon)$, $\psi(t, \varepsilon)$ are obtained if, in the integrals (3) of the unperturbed system (1), one substitutes instead of the solutions of the unperturbed system (1) the solutions of the perturbed system (4). Thus the matter is reduced to determining the functions $c(t, \varepsilon) \equiv P[x(t, \varepsilon), y(t, \varepsilon)]$, $\psi(t, \varepsilon) \equiv \theta[x(t, \varepsilon), y(t, \varepsilon)]$, where $x(t, \varepsilon), y(t, \varepsilon)$ are solutions of (4). In the present work, by means of the method of averaging ([1]), asymptotic approximations are constructed for the functions $\dot{c}(t, \varepsilon)$, $\dot{\psi}(t, \varepsilon)$, making it possible to represent these

functions with arbitrary accuracy with respect to the small parameter ε on a time interval $t \sim 1/\varepsilon$. Knowing approximations for $c(t, \varepsilon)$, $\psi(t, \varepsilon)$, one can obtain approximations for the solutions of (4) by substituting the approximate values $c(t, \varepsilon)$, $\psi(t, \varepsilon)$ into (2).

For the case when the solutions of (1) are purely periodic, a similar problem was considered by us in ([2]). Problems analogous to the one considered here are the subject of works by a number of authors ([3,7,8]); very general results in this direction were obtained in the work of D. V. Anosov. The case considered in the present article is not contained in the works indicated above.

§ 2. **Derivation of asymptotic formulas.** Let, in the domains under consideration, all the functions occurring be continuous, bounded, and continuously differentiable a sufficient number of times, and let all the systems of differential equations occurring satisfy the requirements of existence, uniqueness, and continuous dependence of solutions on initial data and parameters (the function θ may turn out to be multivalued; we shall then assume that the derivatives $\partial\theta/\partial x_i$, $\partial\theta/\partial y_j$ are single-valued and sufficiently smooth functions). Suppose that there exists an open domain G of the variables x, y, at each point of which there begins some trajectory of the system (1), corresponding to the solutions (2), which either lies entirely inside G for $(-\infty < t < \infty)$, or can be continued in both directions up to the boundary of G. We now take a large time interval $[0, a/\varepsilon]$ ($a > 0$ arbitrary), on which we shall consider the solutions (4). Consider now an arbitrary finite open subdomain G_0 of the domain G, lying wholly inside G together with its boundary. We shall consider solutions (4) beginning at $t = 0$ inside G_0. We shall consider these solutions on the whole interval $[0, a/\varepsilon]$, if during this time the trajectory does not leave G_0, or on some interval $[0, b] \subset [0, a/\varepsilon]$ such that for $t \in [0, b]$ the trajectory does not leave G_0. For these solutions (4), on the indicated time intervals, we shall construct asymptotic approximations for $c(t, \varepsilon)$, $\psi(t, \varepsilon)$.

Make in (4) the change of variables by formulas (2), where $c(t, \varepsilon)$, $\psi(t, \varepsilon)$ are the functions of interest to us, or, what is the same, substitute the solutions (4) into (3) and differentiate the expressions obtained by virtue of the system (4); then for $c(t, \varepsilon)$, $\psi(t, \varepsilon)$ we obtain the equations

$$\dot{c} = (\varepsilon f \nabla_x + \varepsilon \varphi \nabla_y) P, \qquad \dot{\psi} = \omega + (\varepsilon f \nabla_x + \varepsilon \varphi \nabla_y) \theta, \tag{5}$$

$$\left(\nabla_x = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\}, \qquad \nabla_y = \left\{ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_m} \right\} \right).$$

The right-hand sides of (5) are expressed, with the aid of (2), through c and ψ and, by the assumptions made, are periodic in ψ with period 2π. Therefore asymptotic approximations for $c(t, \varepsilon)$, $\psi(t, \varepsilon)$ can be obtained by averaging (5) by means of the method ([1]). Thus, discarding in the equation $\dot{c} = (\varepsilon f \nabla_x + \varepsilon \varphi \nabla_y) P$ terms of order of smallness higher than ε and averaging with respect to ψ, we obtain the following equations of the first approximation for c:

$$\dot{c} = \frac{1}{T_0} \int_0^{T_0} (\varepsilon f_0 \nabla_x + \varepsilon \varphi_0 \nabla_y) P \, dt, \tag{6}$$

where the zero subscript here and everywhere below denotes the value of the corresponding function at $\varepsilon = 0$, i.e. $f_0 = f|_{\varepsilon=0}$, $\varphi_0 = \varphi|_{\varepsilon=0}$. (6) describes c with an error $\sim \varepsilon$ on the interval $t \sim 1/\varepsilon$. Since the solutions (2) of system (1) are assumed known and in equations (6) the function c changes slowly (since $\dot{c} \sim \varepsilon$), integration of (6) is in principle simpler than that of (4). Higher approximations for c, ψ, the equations of which, because of their bulkiness, we cannot write out here, are obtained quite

analogously to the method of (1) from equations (5). The integral in (6) is taken along the trajectory (2) of the unperturbed system (1).

§ 3. **Equations containing slowly varying parameters.** Let the unperturbed system have the form

$$\dot{x} = F(x, y, \mu), \qquad \dot{y} = \Phi(x, y, \mu), \tag{7}$$

where $\mu = \{\mu_1, \mu_2, \ldots, \mu_l\}$ is a set of l parameters, and suppose that system (7) satisfies the requirements of §§ 1 and 2, i.e., it has the general solution

$$x = x_0(c, \psi, \mu), \qquad y = T\psi/2\pi + y_0(c, \psi, \mu) \quad (\psi = \omega t + h),$$

where x_0, y_0 are periodic in ψ with period 2π, and to this solution there correspond the integrals

$$c = P(x, y, \mu), \qquad \psi = \theta(x, y, \mu),$$

all functions being periodic in y_j with periods $T_j = \text{const}$. Consider the perturbed system

$$\dot{x} = F(x, y, \mu) + \varepsilon f(x, y, \mu, \varepsilon), \qquad \dot{y} = \Phi(x, y, \mu) + \varepsilon\varphi(x, y, \mu, \varepsilon),$$
$$\dot{\mu} = \varepsilon\alpha(x, y, \mu, \varepsilon), \tag{8}$$

where f, φ, α are periodic in y_j with periods T_j. In (8) the parameters $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ are no longer constant, but vary slowly with the small velocity $\varepsilon\alpha$, which depends on the state of the system. It is required, for the solutions $\mu = \mu(t, \varepsilon), x(t, \varepsilon), y(t, \varepsilon)$ of system (8), to construct asymptotic approximations; and this, analogously to §§ 1 and 2, reduces to the study of the functions

$$\mu = \mu(t, \varepsilon), \quad c(t, \varepsilon) \equiv P[x(t, \varepsilon), y(t, \varepsilon), \mu(t, \varepsilon)],$$

$$\psi = (t, \varepsilon) \equiv \theta[x(t, \varepsilon), y(t, \varepsilon), \mu(t, \varepsilon)].$$

The problem posed reduces to the results of § 2, since to equations (7) one may add the equations $\dot{\mu} = 0$, and to the integrals $c = P$, $\psi = \theta$ add the integrals $\mu = \text{const}$, after which systems (7), (8) and (1), (4) will coincide up to notation. Applying formula (6) to (8), we obtain the first-approximation equations for c, μ:

$$\dot{c} = \frac{1}{T_0}\int_0^{T_0} (\varepsilon f_0 \nabla_x + \varepsilon\varphi_0 \nabla_y + \varepsilon\alpha_0 \nabla_\mu) P\, dt, \qquad \dot{\mu} = \frac{1}{T_0}\int_0^{T_0} \varepsilon\alpha_0\, dt \equiv \overline{(\varepsilon\alpha_0)} \tag{9}$$

$$\left(\nabla_\mu = \left\{\frac{\partial}{\partial\mu_1}, \frac{\partial}{\partial\mu_2}, \dots, \frac{\partial}{\partial\mu_l}\right\}\right).$$

According to (9), in the first approximation μ varies with velocity $\overline{(\varepsilon\alpha_0)}$, equal to the value averaged over a period of the true velocity of its variation.

§ 4. **Systems close to canonical ones.** Consider an unperturbed system of the form

$$\dot{q} = \nabla_p H(p, q, \mu), \qquad \dot{p} = -\nabla_q H(p, q, \mu) \tag{10}$$

$$(q = \{q_1, q_2, \dots, q_n\}, \quad p = \{p_1, p_2, \dots, p_n\}, \quad \mu = \{\mu_1, \mu_2, \dots, \mu_l\},$$

$$\nabla_q = \left\{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n} \right\}, \qquad \nabla_p = \left\{ \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \dots, \frac{\partial}{\partial p_n} \right\})$$

and the corresponding perturbed system

$$\dot{q} = \nabla_p H(p,q,\mu) - \varepsilon f^p(p,q,\mu,\varepsilon), \qquad \dot{p} = -\nabla_q H(p,q,\mu) + \varepsilon f^q(p,q,\mu,\varepsilon),$$
$$\dot{\mu} = \varepsilon\alpha(p,q,\mu,\varepsilon). \tag{11}$$

(10), (11) are a particular case of systems (7), (8). Suppose that (10), (11) satisfy the same requirements as (7), (8), and apply formulas (9). System (10) always has the integral of total energy $E = H(p,q,\mu)$. Applying (9) to this integral, we obtain for (11) the energy equation

$$\begin{aligned}\dot{E} &= \frac{1}{T_0}\int_0^{T_0} (\varepsilon f_0^p \dot{p} + \varepsilon f_0^q \dot{q})\, dt + \frac{1}{T_0}\int_0^{T_0} \varepsilon(\alpha_0 \nabla_\mu) H\, dt \\ &\equiv \frac{1}{T_0}\int_{T_0} (\varepsilon f_0^p\, dp + \varepsilon f_0^q\, dq) + \frac{1}{T_0}\int_0^{T_0} \varepsilon(\alpha_0\nabla_\mu)H\, dt.\end{aligned} \tag{12}$$

(12) means that, in the first approximation, the rate of change of the energy is proportional to the mean power of the disturbing forces εf^p, εf^q over a period, added to the power of the forces causing the change of the parameters μ with velocity $\varepsilon\alpha_0$.

Let us now suppose that, in system (10), the coordinates q_i and momenta p_i $(i = 1, 2, \dots, n)$ can be divided into two groups containing, respectively, m and l $(m + l = n)$ pairs q_i, p_i of canonically conjugate variables, such that in the first group all the momenta are oscillating, while in the second group all the coordinates are oscillating and the momenta are arbitrary. Denote the elements of these groups respectively by q_1, p_1 and q_2, p_2, where q_1, p_1 and q_2, p_2 are, respectively, m-dimensional and n-dimensional vectors. Form the integral

$$I \equiv \int_{t_0}^{t_0+T_0} (p_1\dot{q}_1 - q_2\dot{p}_2)\, dt,$$

where t_0 is arbitrary. In view of the assumptions made, I will not depend on t_0, i.e.

$$I = \int_0^{T_0} (p_1\dot{q}_1 - q_2\dot{p}_2)\, dt = \int_{T_0} p_1\, dq_1 - q_2\, dp_2.$$

The quantity I is called the **action integral**. Applying formula (9) to I, we derive the equation

$$\dot{I} = \int_0^{T_0} (\varepsilon f_0^p \dot{p} + \varepsilon f_0^q \dot{q})\, dt + \int_0^{T_0} \left[\varepsilon\alpha_0 - \overline{(\varepsilon\alpha_0)}\right] \nabla_\mu H\, dt = \tag{13}$$

$$= \int_{T_0} (\varepsilon f_0^p \, dp + \varepsilon f_0^q \, dq) + \int_0^{T_0} \left[\varepsilon\alpha_0 - \overline{(\varepsilon\alpha_0)}\right] \nabla_\mu H \, dt.$$

According to (13), the rate of change of I is proportional to the work of the disturbing forces εf^p, εf^q over a period of oscillations, added to the virtual work of the change of the parameters μ during the period with velocity $\varepsilon\alpha_0 - \overline{(\varepsilon\alpha_0)}$, equal to the deviation of the velocity μ from its mean value.

Let us now suppose that in (11) the parameters μ vary, in the first approximation, uniformly, i.e. $\varepsilon\alpha_0 = \text{const}$, and that the work

$$\int_{T_0} (\varepsilon f_0^p \, dp + \varepsilon f_0^q \, dq)$$

along the trajectory is equal to zero on every interval $\Delta t = T_0$. Then, according to (13), the integral I, with an error of order $\sim \varepsilon$, is conserved during a time $t \sim 1/\varepsilon$: $I = \text{const}$. Integrals conserved under perturbations are called **adiabatic invariants**; thus, I in the case under consideration is an adiabatic invariant. In the physical literature ([4,5]), the invariance of I is known for the case when system (11) is canonical, i.e. when $f^p = \nabla_p H_1$, $f^q = \nabla_q H_1$, where εH_1 is a perturbation of the Hamiltonian function H; the result formulated here is more general.

A number of problems on the asymptotics of solutions of systems similar to (4) are considered in ([3,6–8]).

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.