MATHEMATICS

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1960

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Abstract

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MATHEMATICS

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ON TRANSFORMATION OPERATORS ASSOCIATED WITH SINGULAR GOURSAT PROBLEMS

(Presented by Academician I. G. Petrovskii, 16 X 1959)

Let $L_p^q(\delta)$ $(p, q = 0, 1, 2, \ldots \infty)$ be the set of functions $f(y)$, defined on the segment δ $(0 \le y \le y_0)$ of the y-axis, p times continuously differentiable on δ, and satisfying at the point $y = 0$ the conditions

$$f(0) = f'(0) = f''(0) = \ldots = f^{(q)}(0) = 0. \tag{1}$$

We shall call the **first singular Goursat problem** G_p^q the problem of finding, in the domain D $(0 \le x \le x_0,\ 0 \le y \le y_0)$, those solutions $z(x, y, b)$ of the equation *

$$xz_{xy} + az_x + bz_y = 0 \qquad (a > 0,\ b \ge 0), \tag{2}$$

which are continuous in D together with their derivatives of order p, and assume, for $x = 0,\ y = 0$, the values

$$z(0, y) = f(y), \qquad z(x, 0) = 0, \qquad f(y) \subset L_p^q(\delta). \tag{3}$$

Investigating the dependence of the functions $z(x, y, b)$ on the exponent b of the singular characteristic $x = 0$, we arrive at the theorems:

Theorem 1. *For $b_2 > b_1 \ge 0,\ b = b_2 - b_1,\ p \ge 2,\ q \ge 0$, the functions $z(x, y, b_i)$ $(i = 1, 2)$ are connected by the equality*

$$z(x, y, b_2) = \frac{1}{\Gamma(b)} \left(\frac{a}{x}\right)^b \int_0^y (y - t)^{b-1} e^{-a(y-t)/x} z(x, t, b_1)\, dt, \tag{4}$$

to which, in the case $p = n,\ q \ge 0$, there corresponds the expansion

$$z(x, y, b_2) = \frac{1}{\Gamma(b)} \sum_{k=0}^{n} \frac{1}{k!} \left(-\frac{x}{a}\right)^k \gamma\left(b + k, \frac{ay}{x}\right) D_y^k z(x, y, b_1) + R_n. \tag{5a}$$

Here $\gamma(\alpha, u)$ is Euler' s incomplete gamma-function (²), $D_y^k = \partial^k/\partial y^k$, and, if $\eta = y - \theta t,\ 0 < \theta < 1$, then

$$R_n = \frac{(-1)^{n+1}}{(n+1)!\Gamma(b)} \left(\frac{a}{x}\right)^b \int_0^y t^{n+b} e^{-at/x} D_\eta^{n+1} z(x, \eta, b_1)\, dt. \tag{5b}$$

In the case $p = n,\ q \geq n$, equality (5a) takes the simpler form

$$z(x, y, b_2) = \sum_{k=0}^{n} \frac{\Gamma(b+k)}{k!\Gamma(b)} \left(-\frac{x}{a}\right)^k D_y^k z(x, y, b_1) + R_n. \tag{5c}$$

Theorem 2. *For any* $f(y)$ *of the class* $L_p^q(\delta)$ $(p \geq 2,\ q \geq 0)$, *when* $b_1 > b_2 > 0,\ \bar{b} = b_1 - b_2,\ c_0\Gamma(\bar{b})\Gamma(b_2) = \Gamma(b_1)$, *the following connection formulas hold*

$$z(x, y, b_2) = c_0 \int_0^1 t^{b_2-1}(1-t)^{\bar{b}-1} z(xt, y, b_1)\, dt, \tag{6a}$$

$$z(x, y, b_2) = \sum_{k=0}^{n} \frac{(-x)^k \Gamma(b_1)\Gamma(k+\bar{b})}{k!\Gamma(\bar{b})\Gamma(k+b_1)} D_x^k z(x, y, b_1) + \bar{R}_n. \tag{6b}$$

In this case, putting $\xi = x - \theta t,\ 0 < \theta < 1$, we find

$$\overline{R}_n = \frac{(-1)^{n+1} c_0}{(n+1)!} x^{1-b_1} \int_0^x t^{\bar{b}+n}(x-t)^{b_2-1} D_\xi^{n+1} z(\xi, y, b_1)\, dt. \tag{6c}$$

Let $v(x, y, b)$ denote the solution of the same problem (3) (for $p = 2,\ q = 0$) for the parabolic-type equation $^{(3,4)}$:

$$x v_{xx} + b v_x - a v_y = 0 \qquad (a > 0), \tag{7}$$

whose singular line still coincides with the characteristic $x = 0$.

Theorem 3. Let b_2 not be a positive integer; $0 \leq b_1 < 1$;
$c_1\Gamma(1-b_1)\Gamma(1-b_2) = -1$; and let $K(x, y, \xi, b_1, b_2)$ be the function defined on the interval $0 \leq \xi \leq y$ by the expression

$$K = c_1 e^{a\xi/x} D_\xi \int_\xi^y (y-t)^{b_2-2}(t-\xi)^{-b_1} \exp\left[-\frac{a(x^2 - t^2 + yt)}{x(y-t)}\right] dt. \tag{8}$$

Then $z(x, y, b_1)$ is transformed into $v(x, y, b_2)$ by the relation

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* Equation (2) plays an important role in the theory of confluent hypergeometric functions of several independent variables (¹).

$$v(x,y,b_2)a^{b_1+b_2-1} = x^{1+\bar{b}} \int_0^y Kz(x,\xi,b_1)\,d\xi. \tag{9}$$

Theorem 4. If $c_2\Gamma(b_2) = a^{b_1+b_2}\Gamma(1-b_1)$, $b_2 \geq 0$, and arbitrary $b_1 \neq 1, 2, ...,$

$$z(x,y,b_2) = c_2 \int_0^\infty t^{\frac{b_1+b_2-2}{2}} J_b(2a\sqrt{t})\, v(xt,y,b_1)\,dt. \tag{10}$$

As formulas (4)—(10) show, there exist two types of transformation operators T_x and T_y, acting respectively in the singular and regular variables x and y (5,6). By forming products of two, three, and a larger number of such operators, one can obtain a number of composed connection formulas. Let us note, for example, the multiple transformation

$$z(x,y,b_2) = \int_0^1 \int_0^1 K(\xi,t,b_1,b_2)z(x\xi t,y,b_1)\,d\xi\,dt \qquad (b_2 > 0), \tag{11}$$

in which $\mu > 0$, $\bar{b} > \nu > 0$, $\Gamma(b_1) = c_3\Gamma(b_2)\Gamma(\nu)\Gamma(\bar{b}-\nu)$,

$$K = c_3\xi^{\mu-1}(1-\xi)^{\nu-1}t^{b_2-1}(1-t)^{\bar{b}-\nu-1}F(-\nu, b_1-\nu-\mu, \bar{b}-\nu, 1-t).$$

Let two initial functions $f_1(y)$ and $f_2(y)$ satisfy the given linear integro-differential relation. Considering the corresponding solutions $z_k(x,y,b_s)$ $(k,s = 1,2)$, we obtain other generalizations of the connection formulas found. Thus, if $f_2(y) = P(y)f_1(y)$, where $P(y)$ is an arbitrary function integrable on δ, $b = b_2 - b_1 > 0$,

$$Q = -\exp(a\xi/x)D_\xi J, \qquad \Gamma(b_2)\Gamma(1-b_1)J = \int_\xi^y P(t)(t-\xi)^{-b_1}(y-t)^{b_2-1}\,dt,$$

then

$$z_2(x,y,b_2) = e^{-ay/x}\left(\frac{a}{x}\right)^b \int_0^y Q(x,y,\xi,b_1,b_2)z_1(x,\xi,b_1)\,d\xi. \tag{12}$$

In particular, when $P_1 = (y-y_1)^{-\alpha_1}(y-y_2)^{-\alpha_2}$, $P_2 = (y-y_1)^{-\alpha_1}e^{-ky}$, where the numbers α_1 and α_2 are less than unity, $J(x,y,\xi,b_1,b_2)$ reduce to hypergeometric integrals (1), so that here $J_1 = \bar{P}_1F_1(b_2,\alpha_1,\alpha_2,1+b,Y_1,Y_2)$, $J_2 = \bar{P}_2\Phi_1(b_2,\alpha_1,1+b,Y_1,k(y-\xi))$, $\Gamma(1+b)\bar{P}_\nu = P_\nu(y-\xi)^b$, $(y-y_\nu)Y_\nu = (y-\xi)$ $(\nu = 1,2)$. If one of the quantities α_1, α_2 or k is equal to zero, then F_1 and

Φ_1 are replaced by the hypergeometric functions of Gauss and Kummer. For $P_n = (y-y_1)^{-\alpha_1}(y-y_2)^{-\alpha_2}\cdots(y-y_n)^{-\alpha_n}$ there arise Lauricella functions: $J_n = \bar{P}_{nF}D(b_2,\alpha_1,\alpha_2,\dots,\alpha_n,1+b,Y_1,Y_2,\dots,Y_n)$. For (7), in the analogous case, equality (9) still holds, but the integrand of its kernel (8) now acquires the factor $P(t)$. By the same method one constructs formulas connecting 3 and a larger number of integrals z_k according to a prescribed relation between their initial functions f_k. In the case $f_{n+1} = f_1 f_2 \dots f_n$ $(n = 1,2,\dots)$, multiplication theorems arise for solutions z_k $(k = 1,2,\dots,n+1)$ of the Goursat problem.

If $f(y)$ is an infinitely differentiable function, then formulas (5a), (6b) (for $q \geq 0$) and (5c) (for $q = \infty$), in which the remainder terms R_n tend to zero as $n \to \infty$, may be replaced by infinite series converging absolutely and uniformly in the domain D. Namely, introducing the notation $z_k = z(x,y,b_k)$; $\Delta_y = 1+(x/a)D_y$; $Z_y^b = (ay/x)\gamma^*(b,(ay/x)\Delta_y)$; $V_y^b = \bar{K}_{1-b}(2\sqrt{axD_y})$, we obtain $z_2 = Z_y^b z_1$; $z_2 = \Delta_y^{-b} z_1$; $v_2 = V_y^b Z_y^{1-b_1}\Delta_y z_1$; $v_2 = V_y^{b_2}\Delta_y^{b_1} z_1$. Moreover, (6b) gives

$$z(x,y,b_2) = {}_1F_1(b_1-b_2,b_1,-\delta_x)z(x,y,b_1) \quad (\delta_x = xD_x). \tag{13}$$

For integral negative values of $b = b_2 - b_1$ and $\bar{b} = b_1 - b_2$, the binomial series and the confluent hypergeometric function become finite sums and give recurrence relations: $z(x,y,b) = \Delta_y^m z(x,y,b+m)$, $z(x,y,b+m) = \dfrac{\Gamma(b)m!}{\Gamma(b+m)} L_m^{b-1}(-\delta_x)z(x,y,b)$, where $m = 0,1,2,\dots$, and $L_m^a(u)$ are Laguerre polynomials. It remains to note two more cases of expressing the function ${}_1F_1$, in which $z(x,y,b) = \exp(-\frac{1}{2}\delta_x)\bar{I}_{b-1/2}(\frac{1}{2}\delta_x)z(x,y,2b)$, $z(x,y,1) = \Gamma(b)\gamma^*(b-1,\delta_x)z(x,y,b)$. Solving the integral equations (4), (6a), (9), (10) with respect to $z(x,y,b_1)$ and $v(x,y,b_1)$, we obtain the inverse transformation operators T_x^{-1} and T_y^{-1}. Thus, for example, from (4) and (6a), for $0 < b < 1$, $0 < \bar{b} < 1$, and $\Gamma(b_2) = c_4\Gamma(b_1)\Gamma(1-\bar{b})$, we find

$$z(x,y,b_1) = \frac{1}{\Gamma(1-\bar{b})}\left(\frac{x}{a}\right)^b e^{-ay/x} D_y \int_0^y e^{a\xi/x}(y-\xi)^{-b} z(x,\xi,b_2)\,d\xi, \tag{14a}$$

$$z(x,y,b_1) = c_4 x^{1-b_2} D_x \int_0^x \xi^{b_1-1}(x-\xi)^{-\bar{b}} z(\xi,y,b_2)\,d\xi. \tag{14b}$$

For T_x^{-1} and T_y^{-1} one can also write expansions analogous to those given above for the operators T_x, T_y. In particular, $z(x,y,0) = f(y)$; consequently, for $b_1 = 0$, $b_2 = b$, formulas (4), (5) and equality (9), in which now

$$K = \frac{1}{\Gamma(1-b)}(y-\xi)^{b-2}\exp\left(-\frac{ax}{y-\xi}\right),$$

give the solution of problem (3) for equations (2) and (7). For example, (4) then takes the form

$$z(x, y, b) = \frac{1}{\Gamma(b)} \left(\frac{a}{x}\right)^b \int_0^y (y-t)^{b-1} \exp\left[-\frac{a(y-t)}{x}\right] f(t)\, dt. \qquad (15)$$

Since the integral of equation (2) $z_0 = \frac{1}{\Gamma(b)} \gamma\left(b, \frac{ay}{x}\right)$, corresponding to the value $f(y) = 1$, is contained, for $0 \le y/x \le \infty$, between zero and one,

then from (15) it follows that at no point of the rectangle D does $|z|$ exceed the maximum attained by $|f(y)|$ on the segment δ of the Oy-axis. Thus, the maximum principle, established earlier for hyperbolic equations with regular characteristics (^{7,8}), holds in the case of the singular Goursat problem.

Differentiating (15) with respect to x and y, we find that, if $f'(y) \geqslant 0$ on δ, then $z_x \leqslant 0$, $z_y \geqslant 0$ in the domain D. Therefore in formulas (5) and (6), for $n = 0$, $R_0 \leqslant 0$ and $\bar{R}_0 \geqslant 0$, i.e. $z(x, y, b_2) \leqslant z(x, y, b_1)$, if $b_2 > b_1$ and $(x, y) \in D$. Thus, in the case of a nondecreasing initial function $f(y)$, the derivative $z_b \leqslant 0$ everywhere in D.

If $f(y) = y^m (1-cy)^n e^{ky}$ $(m > 0)$, then, by virtue of (15),

$$z = \frac{\Gamma(1+m)}{\Gamma(1+m+b)} y^{m+b} \left(\frac{a}{x}\right)^b e^{-ay/x} \Phi_1\left(1+m,\, -n,\, 1+m+b,\, cy,\, \frac{ay}{x} + ky\right).$$

On the other hand, for $f(y) = y^m$ $(m > 0)$, for (7) we find

$$v = y^m \left[\frac{\Gamma(b+m)}{\Gamma(b)}\, {}_1F_1\left(-m,\, b;\, -\frac{ax}{y}\right) - \frac{\Gamma(1+m)}{\Gamma(2-b)} \left(\frac{ax}{y}\right)^{1-b} {}_1F_1\left(1-m-b,\, 2-b,\, -\frac{ax}{y}\right)\right].$$

Substituting these values for z and v into the corresponding connection formulas, we obtain a number of integrals with special functions which, in particular cases (when k, m, n, or c are equal to zero), reduce to known addition theorems for the functions $\gamma(\alpha, z)$, $\Gamma(\alpha, z)$, ${}_1F_1(\alpha, \beta, z)$, $\bar{I}_\nu(z)$ (^{2}). Of great interest is also the second singular Goursat problem:

$$z_x(0, y) = f(y), \qquad z(x, 0) = 0 \quad \left(f(y) \subset L_p^q(\delta)\right).$$

Its solution $\bar{z}(x, y, b)$ for (2) has the form

$$\bar{z}(x, y, b) = -\frac{a}{\Gamma(1+b)} \int_0^x \gamma\left[b, \frac{a(y-t)}{x}\right] f(t)\, dt \qquad (-1 < b \neq 0).$$

With the aid of the equality

$$\bar{z}(x, y, b) = -\frac{a}{b} \int_0^y z(x, t, b)\, dt,$$

which connects $\bar{z}$ with z, the results given above for the functions $z(x, y, b)$ carry over to the integrals $\bar{z}(x, y, b)$. Thus, for example, when $b = b_2 - b_1 > -2$,

$$\bar{z}(x, y, b_2) = \frac{x}{a} \int_0^x RD_\xi \left[e^{a\xi/x} z_\xi(x, \xi, b_1)\right] d\xi,$$

where

$$b_2 \Gamma(1 + b) R = b_1 \exp(-a\xi/x)\, \gamma \left[1 + b, \frac{a(y - \xi)}{x}\right].$$

Finally, we note that for solutions $z, \bar{z}$, and v of more general linear equations of hyperbolic and parabolic types with singular characteristics one can construct analogous transformation operators, whose kernels, after the removal of the same multiplicative power and exponential singularities, become continuous bounded or entire holomorphic functions.

Received
15 X 1959

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