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Abstract

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MECHANICS

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ON THE INFLUENCE OF TURBULENT DIFFUSION IN THE WIND DIRECTION ON THE DISTRIBUTION OF THE CONCENTRATION OF A SUBSTANCE DIFFUSING IN THE ATMOSPHERE

(Presented by Academician L. I. Sedov on 15 XII 1959)

1. In the theory of a number of atmospheric processes (turbulent diffusion of a conservative impurity or of moisture, transformation of air masses under the influence of the underlying surface), quantitatively described by the solution of the semiempirical equation of turbulent diffusion

$$\frac{\partial q_0^*}{\partial t} + u\frac{\partial q_0^*}{\partial x} - w\frac{\partial q_0^*}{\partial z} = K_x\frac{\partial^2 q_0^*}{\partial x^2} + K_y\frac{\partial^2 q_0^*}{\partial y^2} + \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial q_0^*}{\partial z}\right) \quad (1)$$

with boundary conditions *

$$q_0^* \to 0 \quad \text{as } \sqrt{x^2+y^2+z^2} \to \infty; \qquad \left[K_z\, \partial q_0^*/\partial z + w q_0^*\right]_{z=z_0} = b q_0^*\big|_{z=z_0} \quad (2)$$

and the initial condition **

$$q_0^*(0,x,y,z) = \delta(x)\delta(y)\delta(z-H), \quad (3)$$

where $q_0^*(t,x,y,z)$ is the volume concentration of the diffusing substance; the Ox axis is directed along the height-averaged constant wind u; the Oz axis is vertically upward; w is the vertical velocity of ordered transport of the substance (gravitational settling); K_x, K_y, K_z are the coefficients of turbulent diffusion, in considering stationary processes the diffusion of the substance in the wind direction is usually neglected, i.e. it is assumed that $K_x = 0$ ([1–3]).

In the present work the validity of this assumption is investigated, i.e. the influence of turbulent diffusion in the direction of the Ox axis on the distribution of

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the volume concentration of the substance $q_1^*(x,y,z)$ is estimated in the case of a unit stationary point source, when

$$q_1^*(x,y,z)=\int_0^\infty q_0^*(t,x,y,z)\,dt,$$

and on the surface concentration $\sigma^*(x,y)$ of the substance absorbed by the earth, in the case of a unit instantaneous point source, when

$$\sigma^*(x,y)=\int_0^\infty \left[K_z\partial q_0^*/\partial z+wq_0^*\right]_{z=z_0}\,dt=\begin{cases} bq_1^*(x,y,z_0), & \text{for } b<\infty,\\ K_z\partial q_1^*/\partial z\big|_{z=z_0}, & \text{for } b=\infty.\end{cases}$$

Assuming K_x and K_y to be constant and introducing dimensionless variables and parameters

$$\zeta=\frac{z}{H};\qquad \zeta_0=\frac{z_0}{H};\qquad \xi=\sqrt{\frac{K_z(H)}{K_x}}\frac{x}{H};\qquad \eta=\sqrt{\frac{K_z(H)}{K_y}}\frac{y}{H};\qquad \tau=\frac{K_z(H)}{H^2}t;$$

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* General boundary condition of partial reflection of the substance from the earth's surface ([1]); for $b=0$ $(b=\infty)$ we obtain the case of reflection (absorption) of the substance by the earth; z_0 is the height of the "roughness layer."

** Condition of a unit instantaneous point source at height H above the earth. The solution corresponding to a continuously acting (stationary) volume source is obviously obtained by integrating the solution of problem (1)-(3) with respect to the spatial coordinates and with respect to time.

$$\nu=\frac{wH}{K_z(H)};\quad \beta=\frac{bH}{K_z(H)};\quad l=\frac{uH}{\sqrt{K_{xK}z(H)}};\quad q_0=q_0^*\frac{\sqrt{K_{xK}y}}{H^3K_z(H)};\quad (4)$$

$$q_1=q_1^*H\sqrt{K_{xK}y};\quad \sigma=\sigma^*\frac{\sqrt{K_{xK}y}}{K_z}H^2,$$

we obtain the solution of problem (1)—(3) in the form

$$q_0(\tau,\xi,\eta,\zeta)=\exp\{[-(\xi-l\tau)^2-\eta^2]/4\tau\}\,g_\nu(\tau,\zeta)/4\pi\tau,$$

where $g_\nu(\tau,\zeta)$ is the solution of the equation

$$\partial g_\nu/\partial\tau-\nu\partial g_\nu/\partial\zeta=\partial\left[K_z\partial g_\nu/\partial\zeta K_z(H)\right]/\partial\zeta,\qquad (5)$$

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satisfying the boundary conditions:

$$g_\nu \to 0 \quad \text{as } \zeta \to +\infty; \qquad [K_z \partial g_\nu / \partial \zeta K_z(H) + (\nu - \beta) g_\nu]_{\zeta=\zeta_0} = 0 \tag{6}$$

and the initial condition $g_\nu(0,\zeta) = \delta(\zeta - 1)$. The expression for the function q_1 takes the form

$$q_1(\xi,\eta,\zeta) = \int_0^\infty \tau^{-3/2} f_\nu(\tau,\eta,\zeta) \exp\{-p(\tau-\tau_0)^2/\tau\}\, d\tau, \tag{7}$$

where

$$p = l^2/4; \qquad \tau_0 = \xi/l; \qquad f_\nu(\tau,\eta,\zeta) = \sqrt{\tau}\, g_\nu(\tau,\zeta) \exp(-\eta^2/4\tau)/4\pi. \tag{8}$$

2. The method for estimating the influence of diffusion in the direction of the Ox axis on the distribution of the concentration q_1 consists in constructing the principal term of the asymptotic expansion of integral (7) as $K \to 0$, i.e., as $p \to \infty$, and in estimating the remainder of this expansion.

The asymptotic expansion of integral (7) as $p \to \infty$ is constructed by Laplace' s method ([4]). Expanding f_ν in the variable τ in a Taylor series of three terms in a neighborhood of the critical point τ_0, after simple transformations we obtain

$$q_1(\xi,\eta,\zeta) = \sqrt{\pi/p}\, f_\nu(\tau_0,\eta,\zeta)\tau_0^{-1} + R_\nu(p,\tau_0,\eta,\zeta) =$$

$$= \sqrt{\pi/p}\, \tau_0^{-1} f_\nu \cdot [1 + r_\nu(p,\tau_0,\eta,\zeta)], \tag{9}$$

where

$$R_\nu(p,\tau_0,\eta,\zeta) = \int_0^\infty \tau^{-3/2} \exp\{-p(\tau-\tau_0)^2/\tau\} \int_{\tau_0}^{\tau} \partial^2 f_\nu\, \partial\tau^{-2} (\tau - t)\, dt\, d\tau.$$

It is easy to see that the principal term of expansion (9) does not contain K_x; therefore r_ν represents the relative error of the volume concentration q_1 that arises when diffusion in the wind direction is neglected. The problem is to find the set D (region) of values of the variables τ_0 (or ξ), η, and ζ for which, at a fixed value of the parameter p (the quantity K), the inequality

$$|r_\nu(p,\tau_0,\eta,\zeta)| \leqslant \varepsilon \tag{10}$$

holds for any prescribed $\varepsilon > 0$.

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In view of the complexity of the expression for r_ν, a direct solution of inequality (10) in the general case is difficult. Therefore we shall construct a simple upper estimate of $|r_\nu|$, assuming that for $\tau \geqslant 0$ the estimate

$$|\varphi_\nu(\tau,\eta,\zeta)| \leqslant C_\nu(\eta,\zeta); \qquad \varphi_\nu(\tau,\eta,\zeta) = \tau^s \partial^2 f_\nu/\partial\tau^2, \quad s \geqslant 0, \tag{11}$$

is valid, and using the estimate of the remainder of the expansion of the Macdonald function $K_\nu(x)$ as $x \to +\infty$ ([5]),

$$|r_\nu(p,\tau_0,\eta,\zeta)| \leqslant$$

$$\leqslant \frac{C_\nu \tau_0^{1-s}}{4pf_\nu(\tau_0,\eta,\zeta)} \left[\sum_{i=0}^{k-1} \frac{\Gamma(s+i)\Gamma(s-2)(4p\tau_0)^{-i}}{\Gamma(s)\Gamma(s-i-2)(i+1)!} + \theta\, \frac{\Gamma(s+k)\Gamma(s-2)(4p\tau_0)}{\Gamma(s)\Gamma(s-k-2)(k+1)!}\right],$$

where $0 < \theta < 1$ for $k \geq s-3$.* Hence, to determine a region D' lying entirely in D, we obtain the inequality:

$$\frac{C_\nu \tau_0^{1-s}}{4pf_\nu(\tau_0,\eta,\zeta)} \sum_{i=0}^{k} \frac{\Gamma(s+i)\Gamma(s-2)(4p\tau_0)^{-i}}{\Gamma(s)\Gamma(s-i-2)(i+1)!} \ll \varepsilon. \tag{12}$$

Putting $k = 1$ ($s \leq 4$), we may write, simplifying this inequality:

$$C_\nu \tau_0^{1-s}/2pf_\nu(\tau_0,\eta,\zeta) \ll \varepsilon \quad \text{for} \quad 8p\tau_0 \geq s(s-3). \tag{13}$$

Let us find the region D' for several frequently occurring particular cases of problem (1)–(3).

3. Consider the case $K_z(z) = \text{const}$, here taking $\zeta_0 = 0$. For a suspended substance ($\nu = 0$), in the cases of reflection ($\beta = 0$) and absorption ($\beta = \infty$) of the substance at ground level, we have

$$f_0(\tau,\eta,\zeta) = [\exp(-\rho_1^2/4\tau) \pm \exp(-\rho_2^2/4\tau)]/8\pi^{3/2}; \qquad \rho_{1,2}^2 = (\zeta \mp 1)^2 + \eta^2,$$

where the upper (lower) sign in the bracket corresponds to the first (second) case. For $s = 3$, estimate (11) will hold, and from inequality (12), for determining the region D', we obtain the inequality

$$\xi^2[\exp(-\rho_1^2 l/4\xi) \pm \exp(-\rho_2^2 l/4\xi)] \geq (1+\eta^2+\xi^2)/\varepsilon, \tag{14}$$

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which on the jet axis ($\rho_1 = 0$) takes the form $\xi^2[1 \pm \exp(-l/\xi)] \geq 2/\varepsilon$, or $\lambda^{-2}(1 \pm e^{-\lambda}) \geq 2/\varepsilon l^2$, where $\lambda = l/\xi$. From the last inequality follows the condition $\xi > \xi_0^{\pm} = l/\lambda_0^{\pm}$, where $\lambda_0^{\pm}$ is the unique positive root of the equation $\lambda^2 = a(1 \pm e^{-\lambda})$; $a = \varepsilon l^2/2$.

The region D' is a cone-shaped body with vertex at the point $P(\xi_0^{\pm}, 0, 1)$, symmetric with respect to the plane $\eta = 0$ and extending to infinity along the jet axis. In the case of reflection of the substance by the ground, part of its boundary in the plane $\zeta = 0$ is the region Ω', bounded by the curve S' with vertex $Q(\xi_1^+, 0, 0)$, where $\xi_1^+ = l/\lambda_1^+$, and λ_1^+ is the unique positive root of the equation $\lambda^2/4 = ae^{-\lambda/4}$. The equation of the curve S' may be written in the form $\eta^2 = 4\mu^+\xi l^{-1} - 1$, where $\xi > \xi_1^+$, and μ^+ is the unique positive root of the equation $\mu = de^{-\mu}$; $d = \varepsilon l\xi/2$.

Example. Taking $\varepsilon = 0.1$; $K_z = 50$ m^2/sec; $K = 10K_z$; $u = 10$ m/sec; $H = 5$ km, we have $l = 10^{2.5}$, $a = 5 \cdot 10^3$. Solving the transcendental equations approximately, we find $\lambda_0^+ \simeq 71$; $\lambda_1^+ \simeq 17$, whence $\xi_0^+ \simeq 4.5$; $\xi_1^+ \simeq 18.6$, and the corresponding values of the dimensional distance $x_0^+ \simeq 14H = 70$ km; $x_1^+ \simeq 59H = 295$ km.**

In the general case $\nu \neq 0$, the solution of problem (5)–(6) has the form [6]

$$g_\nu(\tau, \zeta) = \exp[-(\zeta - 1 + \nu\tau)^2/4\tau]\times$$

$$\times\{1 + e^{-\zeta/\tau} - 2\omega\sqrt{\tau}\, e^{-\zeta/\tau}\Psi[(\zeta + 1 + \omega\tau)/2\sqrt{\tau}]\}/2\sqrt{\pi\tau},$$

where $\omega = 2\beta - \nu > 0$,

$$\Psi(x) = e^{x^2}\int_x^\infty e^{-t^2}\, dt.$$

Constructing $f_\nu(\tau, \eta, \zeta)$, $\partial^2 f_\nu/\partial\tau^2$, and examining their limits as $\tau \to 0$ and as $\tau \to \infty$, we find that estimate (11) holds for any $s > 0$. Taking $s = 4$, we write inequality (13) in the form

$$\tau_0^3 f_\nu(\tau_0, \eta, \zeta) \geq C_\nu/2p\varepsilon; \qquad \tau_0 \geq 1/2p. \tag{15}$$

* For $s = 3$, the expression in square brackets is equal to unity; for $s = 4$, $k = 1$ and $\theta = 1$ exactly.

** In the case under consideration one can obtain an exact finite expression for $r_\nu(p, \tau_0, \eta, \zeta)$. Solving inequality (10) for the numerical values chosen in the example, we find $\xi_1^+ \simeq 6.2$, whence it follows that estimate (12) is not too crude.

It is not hard to see that the left-hand side of the first inequality (15) is bounded above by a quantity independent of ε, and this inequality is not satisfied for fixed ρ for any ε, i.e., the region D' in this case does not exist for sufficiently small ε.

It may be asserted that the region D is also absent here for arbitrarily small ε, since in [6] it is shown that, in the case under consideration ($\nu \neq 0$), the surface concentration σ, as well as the volume concentration q_1, decreases exponentially at a large distance from the source, and the exponent depends on K_x.

Thus, in the case $K_z(z) = \text{const}$ and for a "heavy" substance ($\nu > 0$), in calculating the quantities q_1 or σ one cannot neglect diffusion in the direction of the wind with an arbitrarily small relative error.

4. Consider the case of diffusion in the surface layer of the atmosphere under neutral stratification, i.e., for $K_z(z) = \varkappa v_* z$, $K_z(z)/K_z(H) = \zeta^*$. To simplify the exposition we restrict ourselves to the case $\beta = \nu \geq 0$, when, as is known [7], in the boundary condition (6) one may take $\zeta_0 = 0$, and the solution of the problem (5)–(6) will have the form [1]

$$g_\nu(\tau, \zeta) = e^{-(\zeta+1)/\tau} \zeta^{-\nu/2} I_\nu(2\sqrt{\zeta}/\tau)/\tau,$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. By investigating the limits of f_ν and $\partial^2 f_\nu/\partial\tau^2$ as $\tau \to \infty$ and $\tau \to 0$, taking into account the known properties of $I_\nu(x)$ [5], estimate (11) is established for $s \geq \nu + {}^5/_2$. Choosing $s = \nu + {}^5/_2$, for $\nu \leq {}^3/_2$ ($s \leq 4$), we can write inequality (13) in the following form:

$$\tau_0 \exp[-(\eta^2 + 4\zeta + 4)/4\tau_0](\sqrt{\zeta}/\tau_0)^{-\nu} I_\nu(2\sqrt{\zeta}/\tau_0) \geq 2\pi C_\nu/\rho\varepsilon,$$

$$8\rho\tau_0 \geq (\nu + {}^5/_2)(\nu - {}^1/_2). \tag{16}$$

If $\tau_0 \to \infty$, then the left-hand side of the first inequality (16) increases without bound (as $O(\tau_0)$); i.e., this inequality will be satisfied for any $\varepsilon > 0$ for sufficiently large values of τ_0 (or ξ).

Thus here, in contrast to the case $K_z = \text{const}$ for a "heavy" impurity ($\nu > 0$), the relative error arising from neglecting diffusion in the wind direction in calculating the concentration q_1 will be arbitrarily small at a sufficiently large downwind distance from the source. An analogous result also holds for $\nu > {}^3/_2$ ($s > 4$); only instead of inequality (13) one must use inequality (12), after transforming it somewhat.

The region Ω' for the concentration $q_1(\xi, \eta, 0)$ is determined, in the case $\nu \leq {}^3/_2$, by inequality (16) with $\zeta = 0$. The coordinate ξ_1 of the vertex $Q(\xi_1, 0, 0)$ of the curve S bounding the region Ω' is found from the solution $\lambda_1 = l/\xi$ of the equation

$$\lambda = de^{-\lambda}; \qquad d = \varepsilon l^2/2\psi_\nu(\alpha_\nu), \tag{17}$$

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where

$$\psi_\nu(\alpha) = e^{-\alpha}\{3/4 + 2\nu + \nu^2 + (2\nu+3)\alpha + \alpha^2\}; \qquad \alpha_\nu = \sqrt{\nu + {}^5/_2} - \nu - {}^1/_2,$$

and the equation of the curve S for $\xi > \xi_1$ will have the form

$$\eta^2 + 4 = 4\xi l^{-1} \ln[\varepsilon l \xi / 2\psi_\nu(\alpha_\nu)].$$

Example. Taking $\varepsilon = 0.1$; $H = 50$ m; $u = 5$ m/sec; $\varkappa v_* = 0.1$ m/sec; $K_x = 10K_z(H)$; $\nu = 1$, we find $l = 16$; $y_1 = 0.37$; $\psi_1(y_1) = 3.96$; in equation (17) $d = 3.23$ and $\lambda_1 = 1.09$, whence $\xi_1 = 14.7$ and $x_1 = 46.5H = 2.3$ km.

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* Here $\varkappa = 0.38$ is the von Kármán constant, v_* is the "friction velocity."

Note: Figure translations are in progress. See original paper for figures.

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