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Abstract

Full Text

Mathematics

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On Continuity and Differentiability with Respect to Congruent Sets

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Let $f(x)$ be a real function, taking finite values, defined and measurable on the interval (a, b). Fix some measurable set Q in a neighborhood of zero, having zero as a limit point.

We define the notion of Q-continuity of a function. A function $f(x)$ is called Q-continuous at a point $x \in (a, b)$ if

$$\lim_{h \to 0,\ h \in Q} [f(x+h) - f(x)] = 0$$

(i.e., the limit is considered under the condition that $h \to 0$ while remaining in the set Q).

Analogously one defines the Q-derivative of the function $f(x)$ at the point x:

$$f'_Q(x) = \lim_{h \to 0,\ h \in Q} \frac{f(x+h) - f(x)}{h}.$$

If the set Q is an interval containing zero, then Q-continuity and Q-differentiability are, respectively, ordinary continuity and differentiability.

It is clear that from the ordinary continuity or differentiability of a function at some point x there follows, respectively, Q-continuity or Q-differentiability at this point, whatever Q may be; the converse (for fixed Q) need not hold.

However, if it is assumed that Q-continuity or Q-differentiability of the function $f(x)$ holds at every point of a set $E \subset (a, b)$ of positive measure, then, under certain conditions imposed on Q, it will follow from this that $f(x)$ is ordinarily continuous or differentiable almost everywhere on E.

Here it is assumed that for all $x \in E$ the set Q is one and the same, i.e., continuity or differentiability is considered with respect to congruent sets.

We find a necessary and sufficient condition that must be imposed on the set Q (separately for continuity and differentiability) in order that the stated assertion be true.*

We introduce the following three classes of sets Q having zero as a limit point:

To class (A) we assign those sets Q which have positive measure in every neighborhood of zero, i.e.

$$\operatorname{mes} Q \cdot (-\delta, \delta) > 0$$

for every $\delta > 0$.

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* Some sufficient conditions (under strong restrictions imposed on Q) were obtained by us as a by-product in the paper ([1]) (pp. 399, 411, 428).

In class (B) we place those sets Q that have positive lower density at zero, i.e.

$$\lim_{\delta \to 0+} \frac{\operatorname{mes} Q \cdot (-\delta, \delta)}{2\delta} > 0^*.$$

In class (C) we place those sets Q for which the closure $\overline{Q}$ has positive lower density at zero, i.e.

$$\lim_{\delta \to 0+} \frac{\operatorname{mes} \overline{Q} \cdot (-\delta, \delta)}{2\delta} > 0.$$

Theorem 1. *If a function $f(x)$, measurable on (a, b), is Q-continuous at every point of a set E of positive measure, and the set Q belongs to class (A), then $f(x)$ is continuous in the ordinary sense almost everywhere on the set E.*

For any set Q not belonging to class (A), one can construct a function, measurable on the interval, that will be Q-continuous on some set E of positive measure (and even almost everywhere) and at the same time discontinuous everywhere.

Theorem 2. *If a function $f(x)$, measurable on (a, b), has a finite Q-derivative $f'_Q(x)$ at every point of a set E of positive measure, and the set Q belongs to class (B), then $f(x)$ has an ordinary derivative almost everywhere on the set E.*

For any set Q not belonging to class (B), one can construct a function, measurable on the interval, that has a finite Q-derivative and has no ordinary derivative on some set $\mathcal{E}$ of positive measure.

Theorem 2 can be strengthened by replacing the requirement of Q-differentiability by the requirement of finiteness of the Q-derivative numbers

$$\overline{f}'_Q(x) = \overline{\lim}_{h \to 0,\ h \in Q} \frac{f(x+h) - f(x)}{h}, \qquad \underline{f}'_Q(x) = \underline{\lim}_{h \to 0,\ h \in Q} \frac{f(x+h) - f(x)}{h}.$$

Namely, the following is true.

Theorem 3. *A necessary and sufficient condition for the finiteness of the Q-derivative numbers of a measurable function $f(x)$ on a set E of positive measure to imply ordinary differentiability almost everywhere on E is that the set Q belong to class (B).*

If one considers only functions continuous on (a, b), then the requirements imposed on the set Q can be weakened.

Theorem 4. *If a function $f(x)$, continuous on (a, b), has finite Q-derivative numbers at every point of a set E of positive measure, and the set Q belongs to class (C), then $f(x)$ has an ordinary derivative almost everywhere on E.*

For any set Q not belonging to class (C), one can construct a continuous function that has a finite Q-derivative and has no ordinary derivative on some set $\mathcal{E}$ of positive measure.

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CITED LITERATURE

1. G. Kh. Sindalovskii, *Izv. AN SSSR, ser. matem.*, **22**, No. 3, 395 (1958).

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* Here it is possible that

$$\lim_{\delta\to 0+} \frac{\operatorname{mes} Q \cdot (0, \delta)}{\delta} = 0 \quad \text{and} \quad \lim_{\delta\to 0+} \frac{\operatorname{mes} Q \cdot (-\delta, 0)}{\delta} = 0.$$

Note: Figure translations are in progress. See original paper for figures.

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