# MATHEMATICS

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## Abstract

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MATHEMATICS

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# ASYMPTOTIC EXPANSION OF BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

*(Presented by Academician S. L. Sobolev, 12 XII 1959)*

Consider, in a bounded domain $Q$ of $n$-dimensional space with boundary $\Gamma$, the equation

$$L_\varepsilon u = \varepsilon^2 \Delta\Delta u + L_2 u = h \tag{1}$$

under the boundary conditions

$$u|_\Gamma = 0; \tag{2}$$

$$\Delta u|_\Gamma = 0, \tag{3}$$

where $L_2$ is an elliptic operator of the 2nd order,

$$L_2 = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x), \quad x = (x_1, \dots, x_n),$$

$$c - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i} \geq \alpha^2 > 0. \tag{4}$$

The coefficients of the equation and the boundary of the domain have $N$ derivatives, $h(x) \in W_2^{2N+2}$.

We seek an approximate solution of the problem $A_\varepsilon$ (equation (1) under boundary conditions (2), (3)) for small $\varepsilon$ in the form of a segment of a series in powers of $\varepsilon$.

We shall use the method developed by M. I. Vishik and L. A. Lyusternik ($^1$).

For the boundary-value problem under consideration it is necessary, along with the expansion described below of the operator $L_\varepsilon$, to carry out the same expansion of the boundary operators (2), (3). Near the boundary we introduce a local coordinate system $(\rho, \varphi_1, \dots, \varphi_{n-1})$, where $(\varphi_1, \dots, \varphi_{n-1})$ are coordinates of a point of the boundary, and $\rho$ is the distance along the normal. In equation (1) and in the boundary conditions (2), (3) we pass to the variables $t = \rho/\varepsilon$, $\varphi_1, \dots, \varphi_{n-1}$, and expand the coefficients by Taylor' s formula with respect to the variable $t$. Then

$$L_\varepsilon u \equiv \frac{1}{\varepsilon^2}\left[R_0 + \varepsilon R_1 + \dots + \varepsilon^N R_N + \varepsilon^{N+1} R\right] u, \tag{5}$$

where

$$R_0 \equiv \frac{\partial^4}{\partial t^4} - a^0(\varphi)\frac{\partial^2}{\partial t^2}, \quad \varphi = (\varphi_1, \dots, \varphi_{n-1}), \quad a^0(\varphi) > 0.$$

It is obvious that among the characteristic roots of the equation $R_0 u = 0$ there is a negative one, $-\lambda(\varphi)$, i.e. the degeneration of the problem $A_\varepsilon$ into the problem $A_0$ is regular ($^1$). The $R_k$ are differential operators whose coefficients depend on $t$ only polynomially.

The boundary operators (in the variables $(t, \varphi)$) are written as follows:

$$u\big|_\Gamma = u\big|_{t=0},$$

$$\Delta u\big|_\Gamma = \frac{1}{\varepsilon^2}\left(\frac{\partial^2}{\partial t^2} + \varepsilon l_1 + \varepsilon^2 l_2\right) u\big|_{t=0},$$

where $l_1$ and $l_2$ are differential operators of order 2.

Problem $A_0$, i.e. the equation

$$L_2 w = h \tag{6}$$

with the boundary condition

$$w\big|_\Gamma = C, \tag{7}$$

has a unique solution by virtue of (4).

We construct an approximate solution $\tilde{u}_\varepsilon$ of problem $A_\varepsilon$ in the form $\tilde{u}_\varepsilon = \tilde{u} + \varepsilon^2 \tilde{v}$, where $\tilde{u}$ satisfies (1), and $\varepsilon^2 \tilde{v}$ satisfies the corresponding homogeneous equation, i.e.

$$L_\varepsilon \tilde{u} \equiv \varepsilon^2 \Delta\Delta \tilde{u} + L_2 \tilde{u} = h; \tag{8}$$

$$L_\varepsilon \varepsilon^2 \tilde{v} \equiv \left[R_0 + \varepsilon R_1 + \dots + \varepsilon^{N+1} R\right] \tilde{v} = 0. \tag{9}$$

Fulfillment of the boundary conditions requires for $\tilde{u} + \varepsilon^2 \tilde{v}$:

$$\tilde{u}\big|_\Gamma + \varepsilon^2 \tilde{v}\big|_{t=0} = 0; \tag{10}$$

$$\Delta\tilde{u}\big|_\Gamma + \left(\frac{\partial^2}{\partial t^2} + \varepsilon l_1 + \varepsilon^2 l_2\right)\tilde{v}\big|_{t=0} = 0. \tag{11}$$

We seek $\tilde{u}$ and $\tilde{v}$ in the form

$$\tilde{u} = u_0 + \varepsilon u_1 + \dots + \varepsilon^N u_N + \dots; \qquad \tilde{u}_N = u_0 + \varepsilon u_1 + \dots + \varepsilon^N u_N; \tag{12}$$

$$\tilde{v} = v_0 + \varepsilon v_1 + \dots + \varepsilon^N v_N + \dots; \qquad \tilde{v}_N = v_0 + \varepsilon v_1 + \dots + \varepsilon^N v_N. \tag{13}$$

Substituting (12) and (13) into (8) and (9) and, respectively, into (10) and (11), and collecting terms with equal powers of $\varepsilon$, we obtain equations and boundary conditions for the successive determination of $u_i$ and $v_i$.

Terms for $\varepsilon^0$:

$$L_2 u_0 = h, \qquad u_0\big|_\Gamma = 0.$$

$u_0$ is determined uniquely: $u_0 \equiv w$,

$$R_0 v_0 \equiv \frac{\partial^4 v_0}{\partial t^4} - a^0(\varphi)\frac{\partial^2 v_0}{\partial t^2} = 0; \tag{$14^0$}$$

$$\frac{\partial^2 v_0}{\partial t^2}\bigg|_{t=0} = -\Delta u_0\big|_\Gamma. \tag{$15^0$}$$

We take that solution of this equation which has the character of a boundary layer (1):

$$v_0 = c_0(\varphi)e^{-\lambda(\varphi)t}, \qquad c_0(\varphi) = -\frac{\Delta u_0}{(\lambda(\varphi))^2}.$$

Terms for $\varepsilon^i$:

$$L_2 u_i = -\Delta\Delta u_{i-2}, \qquad u_i\big|_\Gamma = -v_{i-2}\big|_{t=0}.$$

Since $v_{i-2}\big|_{t=0}$ does not depend on $\varepsilon$, $u_i$ also does not depend on $\varepsilon$,

$$R_0 v_i = -R_1 v_{i-1} - R_2 v_{i-2} - \dots - R_i v_0; \tag{$14^i$}$$

$$\frac{\partial^2 v_i}{\partial t^2}\bigg|_{t=0} = -\Delta u_i\big|_\Gamma - l_1 v_{i-1}\big|_{t=0} - l_2 v_{i-2}\big|_{t=0}. \tag{$15^i$}$$

Since $-R_1 v_{i-1} - R_2 v_{i-2} - \dots - R_i v_0$ is a polynomial of degree $2i-1$ in $t$, multiplied by $e^{-\lambda(\varphi)t}$, equation ($14^i$) has a solution of the form

$$v_i = [tS_{2i-1}(t,\varphi) + c_i(\varphi)]e^{-\lambda(\varphi)t},$$

where the coefficients of the polynomial $S_{2i-1}$ are found by the method of selection, $c_i(\varphi)$ from the conditions of satisfaction of the boundary condition ($15^i$), and $v_i$ is defined in the boundary strip along $\Gamma$.

We now multiply $\tilde{v}_N$ by a smoothing function $\psi(t)$, having derivatives of all orders, equal to 1 in some strip along $\Gamma$ and to 0 outside some other, wider strip. We obtain $v_N$, defined in the whole domain.

From the method of constructing the functions $\tilde{u}_N$ and $\tilde{v}_N$ it follows that $\tilde{u}_N + \varepsilon^2 v_N$ satisfies the following equation and boundary conditions:

$$L_\varepsilon(\tilde{u}_N + \varepsilon^2 v_N) = h + \varepsilon^{N+1} G_1; \tag{16}$$

$$(\tilde{u}_N + \varepsilon^2 v_N)\big|_\Gamma = \varepsilon^{N+1} g_1; \tag{17}$$

$$\Delta(\tilde{u}_N + \varepsilon^2 v_N)\big|_\Gamma = \varepsilon^{N+1} g_2, \tag{18}$$

where the functions $G_1, g_1, g_2$ are of order $O(1)$ with respect to $\varepsilon$.

Let $\alpha$ be a function of order $\varepsilon^{N+1}$ satisfying (17) and (18). Then the function $z$

$$z = u_\varepsilon - (\tilde{u}_N + \varepsilon^2 v_N - \alpha)$$

satisfies the equation

$$L_\varepsilon z = \varepsilon^{N+1} G; \qquad G = O(1) \tag{19}$$

and the homogeneous boundary conditions (2), (3).

Consider the quadratic form:

$$(L_\varepsilon z, z) = \varepsilon^2 (\Delta\Delta z, z) + (L_2 z, z) =$$

$$= \varepsilon^2 \iint_Q \left[\sum_{i=1}^n \frac{\partial^2 z}{\partial x_i^2}\right]^2 d\Omega + \iint_Q \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\, d\Omega + \iint_Q \left[c - \frac{1}{2}\sum_{i=1}^n \frac{\partial b_i}{\partial x_i}\right] z^2\, d\Omega. \tag{20}$$

The integrals over the boundary of the domain $\Gamma$ are equal to 0, since $z$ satisfies the homogeneous boundary conditions (2), (3).

From the last equality (20) we obtain:

$$\varepsilon^2 \iint_Q \sum_{i,j=1}^n \left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)^2 d\Omega + \iint_Q \sum_{i=1}^n \left(\frac{\partial z}{\partial x_i}\right)^2 d\Omega + \iint_Q z^2\, d\Omega \leq$$

$$\leq M\left|(\varepsilon^{N+1}G, z)\right| \leq M\varepsilon^{N+1}\left\{\iint_Q G^2\,d\Omega\right\}^{1/2}\left\{\iint_Q z^2\,d\Omega\right\}^{1/2}.$$

In deriving this inequality we have used the ellipticity of the operator $L_2$, inequality (4), and the inequality

$$\iint_Q \sum_{i,j=1}^{n}\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)^2 d\Omega \leq C_1 \iint_Q \left(\sum_{i=1}^{n}\frac{\partial^2 z}{\partial x_i^2}\right)^2 d\Omega$$

(see (2), Ch. II).

Thus we obtain estimates in $\mathcal{L}^2$ for $z$, its first and second derivatives:

$$\left\{\iint_Q z^2\,d\Omega\right\}^{1/2} \leq C\varepsilon^{N+1}; \tag{21}$$

$$\left\{\iint_Q \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 d\Omega\right\}^{1/2} \leq C\varepsilon^{N+1}; \tag{22}$$

$$\left\{\iint_Q \sum_{i,j=1}^{n}\left(\frac{\partial^2 z}{\partial x_i \partial x_j}\right)^2 d\Omega\right\}^{1/2} \leq C\varepsilon^{N}. \tag{23}$$

We obtain estimates of the third and fourth derivatives from Lemma 1, ($^2$) Ch. II:

$$\sum_{s=0}^{4}\iint_Q \left(\sum_{\alpha_1,\ldots,\alpha_j}\frac{\partial^s z}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_j}}\right)^2 d\Omega \leq C'\left[\iint_Q z^2\,d\Omega + \iint_Q (\Delta z)^2\,d\Omega + \iint_Q (\Delta\Delta z)^2\,d\Omega + \iint_Q \sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 d\Omega + \right. \tag{24}$$

Since

$$\varepsilon^2 \Delta\Delta z = -L_2 z + \varepsilon^{N+1} G, \tag{25}$$

it follows that

$$\varepsilon^2 \iint_Q (\Delta\Delta z)^2\,d\Omega \leq C\iint_Q (L_2, z)^2\,d\Omega + \varepsilon^{2N+2}\iint_Q G^2\,d\Omega,$$

$$\iint_Q (\Delta\Delta z)^2 \, d\Omega \leq C\varepsilon^{2N-2}.$$

We obtain the estimate for the last integral in (24) by multiplying (25) by $\Delta z$ and taking into account that $\Delta z|_\Gamma = 0$,

$$\iint_Q \sum_{i=1}^{n} \left( \frac{\partial}{\partial z}(\Delta z) \right)^2 \, d\Omega \leq C\varepsilon^{2N-2}.$$

From (24) we obtain the estimates

$$\left\{ \iint_Q \sum_{i,j,k=1}^{n} \left( \frac{\partial^3 z}{\partial x_i \partial x_j \partial x_k} \right)^2 d\Omega \right\}^{1/2} \leq C\varepsilon^{N-1}; \tag{26}$$

$$\left\{ \iint_Q \sum_{i,j,k,l=1}^{n} \left( \frac{\partial^4 z}{\partial x_i \partial x_j \partial x_k \partial x_l} \right)^2 d\Omega \right\}^{1/2} \leq C\varepsilon^{N-1}. \tag{27}$$

**Remark.** For the equation

$$L_\varepsilon u = \varepsilon^2 \Delta\Delta u + L_2 u = h,$$

where

$$L_2 u = -\Delta u + \sum_{i=1}^{n} b_i(x_1, \dots, x_n) \frac{\partial u}{\partial x_i} + c(x_1, \dots, x_n) u$$

under the boundary conditions

$$\left. \frac{\partial u}{\partial n} \right|_\Gamma = 0, \qquad \left. \frac{\partial}{\partial n}(\Delta u) \right|_\Gamma = 0$$

there is an analogous representation of the solution $u = \tilde{u} + \varepsilon^3 \tilde{v}$, where

$$\tilde{u} = u_0 + \varepsilon u_1 + \dots + \varepsilon^N u_N + \dots,$$

$$\tilde{v} = v_0 + \varepsilon v_1 + \dots + \varepsilon^N v_N + \dots$$

For the residual $z$ and its derivatives we obtain the same estimates (21), (22), (23), (26), (27).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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