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Abstract

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EMBEDDING THEOREMS FOR A DOMAIN WITH ZERO ANGULAR POINTS

(Presented by Academician S. L. Sobolev on 13 I 1960)

S. L. Sobolev' s embedding theorems were proved by him for domains that are the sum of a finite number of domains star-shaped with respect to a ball ([1]). V. P. Il' in considered domains each boundary point of which is reachable by a fixed straight circular cone lying entirely in the domain. In the present work we obtain embedding theorems for domains whose boundaries have angular points with zero angles, i.e., points that are not reached by any cone.

The method we use is close to V. P. Il' in' s method.

1. We shall call a closed n-dimensional domain V_n, with parameters a, α_0, λ, bounded by surfaces whose equations in a given rectangular coordinate system have the form

$$x_2^2 + \cdots + x_n^2 = \alpha_0^2 x_1^{2\lambda}, \qquad x_1 = a \qquad (x_1 \geq 0,\ \lambda \geq 1).$$

Lemma. Suppose that in the domain V_n a function is given that has continuous partial derivatives up to order l inclusive. Then the inequality

$$|f(0)| \leq \frac{N_1}{a^{\lambda(n-1)+1}} \int_{V_n} |f(Q)|\, dQ + N_2 \int_{V_n} \frac{|D^l f|}{r^{\lambda(n-1)+1-l}}\, dQ, \tag{1}$$

holds, where

$$|D^l f| = \left(\sum_{i_1,\ldots,i_l=1}^{n} \left| \frac{\partial^l f}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^2 \right)^{1/2};$$

r is the distance from the point Q to the vertex of the conical body; the constants N_1 and N_2 do not depend on the choice of the function f for fixed α_0, λ, n, and l, and are uniformly bounded on every finite interval of variation of the parameter a.

2. As usual, by $W_p^{(l)}$ we shall denote the set of functions summable in an n-dimensional domain Ω, obtained by closing the set of $C^{(l)}$ functions having in Ω continuous partial derivatives up to order l, with respect to the norm

$$\|f\|_{W_p^{(l)}} = \|f\|_{L_p} + \|D^l f\|_{L_p}.$$

Theorem 1. Suppose that every boundary point of the domain Ω is reachable by a body congruent to a fixed conical body V_n, with parameters a, α_0, λ, and lying entirely in $\overline{\Omega}$. If

$$n < \frac{pl-1}{\lambda} + 1,$$

then every function $f \in W_p^{(l)}$ is continuous in $\overline{\Omega}$, and for it the inequality

$$|f(p)| \leq \frac{C'}{\varepsilon^{\frac{\lambda(n-1)+1}{p}}} \|f\|_{L_p} + C'' \varepsilon^{l - \frac{\lambda(n-1)+1}{p}} \|D^l f\|_{L_p}, \tag{2}$$

holds, where $0 < \varepsilon \leq a$; C' and C'' depend on the numbers α_0, λ, n, l, and p.

Replacing on the right in inequality (2) the norm $\|D^l f\|_{L_p}$ by $\|f\|_{W_p^{(l)}}$ and taking the minimum with respect to ε, we can write this inequality in another equivalent form

$$\|f\|_C \leq K \|f\|_{L_p}^{1-\varkappa} \|f\|_{W_p^{(l)}}^{\varkappa}, \tag{2'}$$

where

$$\varkappa = \frac{\lambda(n-1)+1}{lp}.$$

3. To obtain the embedding theorem into the space L_p, the following conditions are imposed on the domain considered above:

a) on the boundary there is a finite number N of points which are not attainable by a right circular cone;

b) for each such point there is a neighborhood K_i $(i = 1, \dots, N)$ such that in the domain $K_i \cap \overline{\Omega}$ any point is attainable by means of a parallel displacement of a certain fixed position of a conical body;

c) each point of the domain

$$\overline{\Omega} - \sum_{i=1}^{N} K_i \cap \overline{\Omega}$$

is attainable by a right circular cone with fixed aperture angle and height equal to a.

Theorem 2. *Let the domain* Ω *satisfy conditions a), b), c). If*

$$n > \frac{pl-1}{\lambda} + 1,$$

then the space $W_p^{(l)}$ *is embedded in* L_{q^*}*, where*

$$p \le q^* < q = \frac{p[\lambda(n-1)+1]}{\lambda(n-1)+1-pl},$$

and for every $f \in W_p^{(l)}$ *the inequality*

$$\|f\|_{L_{q^*}} \le \frac{M_1}{\varepsilon^{\frac{\lambda(n-1)+1}{p} - \frac{\lambda(n-1)+1}{q^*}}} \|f\|_{L_p} + M_2 \varepsilon^{l - \lambda\frac{(n-1)+1}{p} + \frac{\lambda(n-1)+1}{q^*}} \|D^l f\|_{L_p}, \qquad (3)$$

holds, where $0 < \varepsilon \le a$*;* M_1 *and* M_2 *depend only on the numbers* a_0, λ, n, l, p.

Analogously to how (2′) is obtained from (2), from inequality (3) follows the inequality

$$\|f\|_{L_{q^*}} \le K_1 \|f\|_{L_p}^{1-\varkappa_1} \|f\|_{W_p^{(l)}}^{\varkappa_1},$$

where

$$\varkappa_1 = \frac{\lambda(n-1)+1}{lp} - \frac{\lambda(n-1)+1}{lq^*}.$$

4. For $\lambda = 1$, our theorems yield, as a special case, the theorems of S. L. Sobolev. For $\lambda > 1$, the theorems of S. L. Sobolev, without changing the formulations, are no longer valid.

Indeed, consider an n-dimensional domain which, in a given rectangular coordinate system, is bounded by surfaces with equations

$$x_2^2 + \cdots + x_n^2 = a_0^2 x_1^{2\lambda}, \qquad x_1 = a \quad (0 \le x_1 \le a \le 1,\ \lambda > 1,\ n \ge 2),$$

and the function defined on it

$$f = \left[x_1^{\frac{\lambda(n-1)-1}{2}} \ln x_1\right]^{-1};$$

it is not difficult to show that $f \in W_2^{(1)}$. By our embedding theorem, $f \in L_{q^*}$, where

$$q^* < q = \frac{2[\lambda(n-1)+1]}{\lambda(n-1)-1}.$$

Further, one can verify that $f \notin L_{q+\varepsilon}$, where $\varepsilon > 0$.

It follows from this that, for the domain under consideration, Theorem 2 cannot be strengthened, and at the same time this means that the S. L. Sobolev embedding theorem for this domain does not hold.

5. Let a conical body be given such that, in a given rectangular coordinate system, it is bounded by surfaces with equations

$$x_2^2 + \cdots + x_n^2 = a_0^2 \left(\frac{x_1}{\ln x_1} \right)^2$$

and $x_1 = a < 1$. Let Ω be a domain each point of which is attainable by a congruent conical body. Such a domain

may have angular points with zero angles on its boundary. It can be shown that, for it, the embedding theorems remain valid without changing the formulations (1).

6. In the work of V. P. Glushko and S. G. Krein (^{2}) it is shown how, from inequalities of the type (2) and (3), one can draw conclusions about the properties of fractional differential operators. The corresponding conclusions can also be drawn in our case. At the same time, the presence on the boundary of the domain of angular points with zero angles can substantially change the properties of fractional powers of operators.

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[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.
[2] V. P. Glushko, S. G. Krein, *DAN*, **122**, No. 6 (1958).

Note: Figure translations are in progress. See original paper for figures.