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Abstract

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INVARIANT HYPERBOLIC SYSTEMS AND THE GOURSAT PROBLEM

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In the paper ([1]) a class of systems was described that admit consideration on an arbitrary Riemannian manifold. It was noted there that, for a definite metric tensor, the systems are elliptic, while for a metric tensor with Lorentz signature they are hyperbolic. In ([2]) the simplest problems for elliptic systems were considered. We now wish to dwell on the structure of the hyperbolic case, to indicate the formulation of the Cauchy problem and of the mixed problem, and to consider in somewhat greater detail the Goursat problem, the possibility of posing which is a specific feature of the systems under investigation.

The possibility of specifying a Lorentz metric on a manifold is equivalent ([3]) to the possibility of specifying a continuous vector field. In the presence of such a field it makes sense to consider coordinate systems with a distinguished coordinate– "time." In local coordinates $x^1, \dots, x^n$ we shall always take such a distinguished coordinate to be $x^n \equiv t$. In this case the index n of every tensor also turns out to play a special role. In accordance with ([2]), we shall call by the covariant $\overset{p}{\omega}$ of a system the essentially different coefficients of a differential form of degree p. In the presence of the distinguished index n, or, equivalently, restricting ourselves to transformations that leave the direction x^n invariant, $\overset{p}{\omega}$ naturally decomposes into two covariants $\overset{p}{u}$ and $\overset{p-1}{u}$, considered on a manifold of dimension $n-1$ and depending on t as on a parameter. Here $\overset{p}{u}$ contains those components of ω numbered by collections of indices not containing the index n, while $\overset{p-1}{u}$ contains the remaining components. Denoting by ∂ the operation of differentiation $\partial/\partial x^n$, one can carry out the corresponding splitting of the exterior differentiation operator d and of the metrically adjoint operator δ (assuming that the metric tensor in the chosen coordinate system satisfies the additional condition $g_{in} = 0$, $i = 1, \dots, n-1$; $g_{nn} = -1$).

After this the invariant system (the system (K_n^*) in ([1])) is written in the form

$$
\begin{array}{ll}
d\overset{0}{u} + \delta\overset{2}{u} - \partial\overset{1}{u} = \overset{1}{f}, & \delta\overset{1}{u} + \partial\overset{0}{u} = \overset{0}{f}, \\
d\overset{2}{u} + \delta\overset{4}{u} - \partial\overset{3}{u} = \overset{3}{f}, & d\overset{1}{u} + \delta\overset{3}{u} + \partial\overset{2}{u} = \overset{2}{f}, \\
\dots\dots\dots\dots & \dots\dots\dots\dots \\
d\overset{n-4}{u} + \delta\overset{n-2}{u} - \partial\overset{n-3}{u} = \overset{n-3}{f}; & d\overset{n-3}{u} + \delta\overset{n-1}{u} + \partial\overset{n-2}{u} = \overset{n-2}{f}, \\
d\overset{n-2}{u} - \partial\overset{n-1}{u} = \overset{n-1}{f}; &
\end{array}
\tag{T}
$$

where d, δ act with respect to the variables $x^1, \dots, x^{n-1}$. Let us note at once that the formally adjoint system of equations, written briefly as

$$T^* v = g, \tag{T*}$$

is obtained from (T) by changing the sign before all operators ∂ to the opposite one.

Lemma 1. Every covariant satisfying the homogeneous system (T) satisfies a system of second order of the form

$$(d\delta + \delta d)\, {}^p u + \partial^2\, {}^p u = 0 \qquad (p = 0, 1, \dots, n-1). \tag{1}$$

Since the first of the operators in (1) is the Laplacian taken with the opposite sign, in Cartesian coordinates system (1) splits into wave operators applied to each of the unknown functions.

Taking as our immediate task the consideration of invariant systems in the simplest domains of Euclidean space, we shall restrict ourselves to a very special choice of the manifold Q—the domain of definition of the system (T)—assuming that it has the form

$$Q = M \times l, \tag{2}$$

where M is a compact $(n-1)$-dimensional Riemannian manifold without boundary, and l is the interval $[0, 1]$. We shall not regard the manifold Q as Riemannian, contenting ourselves with the existence of a (definite) metric on M. On Q, the coordinate t (defined by the position of a point in l) will play the role of a parameter. Understanding by u the totality of all covariants ${}^p u$ $(p = 0, 1, \dots, n-1)$, we define on Q the scalar product and norm by setting

$$(u, u) = \int_0^1 \sum_1^{n-1} ({}^p u, {}^p u)\, dt, \qquad |u, H|^2 = (u, u), \tag{3}$$

where under the integral is the sum of the scalar products taken on M. Completing, with respect to the introduced norm, the set of smooth covariants, we obtain the Hilbert space H. On covariants subject to one of the initial conditions

$$u\big|_{t=0} = 0; \qquad (\;)$$

$$v\big|_{t=1} = 0, \qquad (\Gamma^*)$$

one defines in the usual way ([4]) the strong (by means of an approximating sequence) and weak (using the equality $(u, Tv) = (f, v)$) extensions of the operators T, T^*. All subsequent assertions will be formulated for T, with their automatic validity for T^* understood.

Lemma 2. The weak and strong extensions of the operator T are equivalent.

A covariant u satisfying system (T), in which the operator is understood in the sense of the extension introduced, will be called a **generalized solution** of the system.

Lemma 3. For a generalized solution of system (T) the inequality

$$|u, H| \leqslant 2\,|Tu, H| \qquad (\;)$$

holds.

From Lemmas 2 and 3 the following theorem follows:

Theorem 1. A generalized solution of system (T), for any right-hand side from H, exists and is unique.

The mixed problem in a domain of Euclidean space. Let system (T) be given in a domain G of the form $G = V \times l$, where V is an $(n-1)$-dimensional domain of Euclidean space, homeomorphic to a ball, with sufficiently smooth boundary. Considering a second copy of the domain V and identifying the corresponding boundary points (see ([2])), we complete V to a closed manifold M, and the domain G to a manifold of the form (2). Passing, as in ([2]), to the consideration of the subspaces of even and odd covariants, we obtain from Theorem 1 the theorem on existence and uniqueness of the generalized solution of the mixed problem under the initial conditions (Γ)

and conditions on the lateral surface G, corresponding to the boundary conditions for the elliptic systems (K_{n-1}), (K^*_{n-1}), described in (2).

The Cauchy problem in a domain of Euclidean space.
Let the right-hand sides of the system (T) be given in a domain G of the type under consideration. Since the right-hand sides are required only to belong to H, we may regard V as an $(n-1)$-dimensional cube, extending f, if necessary,

by zero. Identifying the opposite sides of V, we turn the cube into a closed manifold M, homeomorphic to a torus, with the metric induced by the metric in V. Between a manifold Q of the form (2) and the domain G there exists a natural one-to-one correspondence, isometric on each section $t = \text{const}$. Defining in the corresponding way the generalized solution of the Cauchy problem, we obtain from Theorem 1 the theorem on existence and uniqueness of the generalized solution.

The Goursat problem. Let a domain G of Euclidean space be bounded by two cones

$$(x^1)^2 + \cdots + (x^{n-1})^2 - (x^n \pm 1)^2 = 0, \qquad -1 \le x^n \le 1, \tag{S}$$

intersecting in the unit $(n-1)$-dimensional sphere of the plane $x^n = 0$. We shall first study the characteristic matrices of invariant systems (in an orthonormal basis) in the elliptic and hyperbolic cases. Denote by A_n the characteristic matrix in the elliptic case (system (K_n^*) in (1)) and by B_n in the hyperbolic case (system (T)). These matrices have the form (differentiation $\partial/\partial x^k$ corresponds to the variable ξ_k):

$$A_n = \begin{pmatrix} A_{n-1} & \xi_n E \\ \xi_n E & -A_{n-1} \end{pmatrix}, \qquad B_n = \begin{pmatrix} A_{n-1} & -\xi_n E \\ \xi_n E & -A_{n-1} \end{pmatrix}, \tag{4}$$

i.e., they split into four blocks of order 2^{n-2} of the indicated form (E is the identity matrix). For $n = 2$,

$$A_2 = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix};$$

the subsequent matrices are determined by induction. If $\widehat{A}_n, \widehat{B}_n$ denote the matrices obtained from the matrices (4) by replacing ξ_k by $\cos \nu x^k$, where ν is the exterior normal to the surface (S), then Green' s formula for the system (T) is written in the form

$$(Tu, v) = \int_S [\widehat{B}_n u, \overline{v}]\, ds + (u, T^* v), \tag{5}$$

where the square brackets denote the sum of pairwise products of the elements of the rows $\widehat{B}_{nu}$ and $\overline{v}$; the bar over v indicates that the elements of the row $\overline{v}$ are numbered in a somewhat different order (as follows from the way the system (T) is written) than the elements of the column u.

Lemma 4. *On the surface (S) the last 2^{n-2} rows of the matrix $\widehat{B}_n$ are linear combinations of the first 2^{n-2} rows (which are independent). If the covariants forming v are chosen so that $\overline{v} = (-1)^p u^p$, then the quadratic form in the surface* *integral of formula* (5) *is nonpositive on the lower half* S_1 *(on which* $x^n \leq 0$*) of the surface* (S) *and nonnegative on the upper half* S_2 *(where* $x^n \geq 0$*).*

If we now define the generalized solution of the system (T) under the boundary conditions

$$(\widehat{A}_{n-1}, E)u\big|_{S_1} = 0 \tag{6}$$

(the Goursat problem), where the written equality means the vanishing on S_1 of 2^{n-2} linear combinations obtained by multiplying the rows of the matrix $(\widehat{A}_{n-1}, E)$ by the column of 2^{n-1} components u, then Lemma 4 makes it possible to obtain an inequality of the form (Φ) for these solutions and for

solutions of the adjoint problem (corresponding to boundary conditions of the form (6), with S_1 replaced by S_2). Hence the theorem follows in the usual way:

Theorem 2. *The generalized solution of the Goursat problem for the system (T) exists and is unique.*

Remark. Using (4), it is not difficult to compute the characteristic determinants of the invariant systems (in the Euclidean case). They have the form $(\xi_1^2 + \cdots + \xi_{n-1}^2 \pm \xi_n^2)^{2^{n-2}}$, where the plus sign corresponds to the elliptic case, and the minus sign to the hyperbolic case.

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Note: Figure translations are in progress. See original paper for figures.