MATHEMATICS

V. F. KOROP

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. F. KOROP

THE INVERSE SCATTERING PROBLEM FOR EQUATIONS WITH A SINGULARITY

(Presented by Academician V. I. Smirnov on 4 II 1960)

1. In the present paper we solve the inverse problem of scattering theory for the boundary-value problem defined by the equation

$$y'' + \left[V(x) + \frac{n(n+1)}{x^2}\right] y + \lambda^2 y = 0 \qquad (0 < x < \infty), \tag{1}$$

where $n = -1, 0, 1, 2, \ldots$, and by the boundary condition

$$\begin{array}{rl} y(0) = 0 & \text{for } n \neq -1, \\ W[y_1(x), y(x)]_{x=0} = (y_1 y' - y_1' y)_{x=0} = 0 & \text{for } n = -1^*. \end{array} \tag{2}$$

Here $y_1(x)$ is some solution of equation (1) for $n = -1$ and $\lambda = 0$. We assume the potential $V(x)$ to be real and to satisfy, for some $\alpha > 0$, the condition

$$\int_0^\infty x^{1+\theta} \, |V(x)| \, dx < \infty, \qquad -\alpha < \theta < \alpha. \tag{3}$$

The investigation is carried out by a method analogous to that used in the works of Z. S. Agronovich and V. A. Marchenko ([1,2]).

We shall call problem (1), (2) **regular** for $n = -1, 0$, and a **problem with a singularity** for $n \geq 1$.

2. The regular problem has a continuous spectrum on the positive half-axis $\lambda^2 > 0$ and a finite number of negative eigenvalues (simple) $\lambda_k^2 < 0$ ($k = 1, 2, \ldots, p$). There exist solutions $u(x, \lambda)$ of the problem which give rise to Parseval' s equality, equivalent to the expansion of the δ-function

$$\delta(x - t) = \sum_{k=1}^{p} u(x, \lambda_k) \, \overline{u(t, \lambda_k)} + \frac{1}{2\pi} \int_0^\infty u(x, \lambda) \, \overline{u(t, \lambda)} \, d\lambda,$$

and which satisfy, as $x \to \infty$, the asymptotic equalities

$$u(x,\lambda) \sim e^{i\lambda x} + (-1)^{n+1} s(-\lambda) e^{-i\lambda x} \qquad (\lambda^2 > 0),$$
$$u(x,\lambda_k) \sim M_k e^{-|\lambda_k| x}, \tag{4}$$

where M_k are positive numbers; $s(\lambda)$ is the scattering function, possessing the properties

$$|s(\lambda)| = 1, \qquad s(-\lambda)s(\lambda) = 1 \qquad (-\infty < \lambda < \infty).$$

* In the case $y_1(0) = 0$ the condition $W[y_1(x), y(x)]_{x=0} = 0$ becomes the condition $y(0) = 0$; in the case when the potential $V(x)$ is summable in a neighborhood of zero and $y_1(0) \neq 0$, it becomes the condition $y'(0) + hy(0) = 0$.

All that has been said remains valid also for the problem with a singularity ($n \geqslant 1$), with the sole difference that one of the eigenvalues, say λ_p, may be equal to zero; the corresponding eigenfunction $u(x, \lambda_p)$ in this case, as $x \to \infty$, satisfies, instead of (4), the asymptotic equality

$$u(x,\lambda_p) \sim M_p x^{-n} \qquad (\lambda_p = 0).$$

The collection of quantities $s(\lambda)$, λ_k, M_k will be called the **scattering data**.

3. Theorem. *In order that the function $s(\lambda)$, $|s(\lambda)| = 1$, $s(-\lambda)s(\lambda) = 1$, and the numbers $\lambda_k^2 < 0$, $M_k > 0$ $(k = 1, 2, \dots, p)$ be the scattering data of some regular problem $(n = -1$ or $0)$ with potential $V(x)$ satisfying inequality (3), it is necessary and sufficient that the following conditions hold:*

1) *There exists a function*

$$f_s(x) = \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} [1 - s(\lambda)] e^{i\lambda x} \, d\lambda, \tag{5}$$

and $f_s(x) \in L(-\infty, \infty)$; for $x > 0$ there exists $f_s'(x)$, and $x^{1+\theta} f_s'(x) \in L(0, \infty)$, if $-\alpha < \theta < \alpha$.

2) *The number of eigenvalues*

$$p = \frac{1}{2\pi}[\eta(0) - \eta(\infty)] + \frac{(-1)^n}{4}[s(0) - 1],$$

where $\eta(\lambda) = \arg s(\lambda)$.

Theorem 2. *In order that the function $s(\lambda)$, $|s(\lambda)| = 1$, $s(-\lambda)s(\lambda) = 1$, and the numbers $\lambda_1^2 < \lambda_2^2 < \dots < \lambda_p^2 \leqslant 0$, $M_k > 0$ $(k = 1, 2, \dots, p)$ be the scattering* *data of some problem with a singularity* ($n \geqslant 1$) *and potential* $V(x)$ *satisfying inequality (3), it is necessary and sufficient that the following conditions hold:*

1) *This condition is formulated exactly as condition 1) of Theorem 1.*

2) *The number of eigenvalues*

$$p = \frac{1}{2\pi}[\eta(0) - \eta(\infty)], \tag{6}$$

where $\eta(\lambda) = \arg s(\lambda)$.

3) $s(0) = 1$.

From Theorems 1 and 2 the following corollary follows: the scattering function $s(\lambda)$ of any problem with a singularity is at the same time the scattering function of some regular problem with the zero boundary condition ($n = 0$), i.e. of the simplest problem.

The converse assertion is, generally speaking, false, since in the case of a regular problem the equality $s(0) = 1$, whose fulfillment is required by condition 3) of Theorem 2, need not hold. Another case is also possible, when $s(0) = -1$.

Let us note that the principal difficulties in the proof of these theorems concern the case when the solution $y_1(x)$ of equation (1) for $\lambda = 0$, satisfying the boundary condition (2), has the asymptotic behavior*

$$y_1(x) \sim Cx^{-n} \qquad (x \to \infty).$$

These difficulties also occurred in work (1), which contains, in particular, the results of the present note for $n = 0$.

---

* For the regular problem this means that $s(0) = -1$; for the problem with a singularity, that there is a zero eigenvalue.

4. We now give an algorithm for reconstructing the potential from the scattering data. By formula (5) we find $f_s(x)$, construct the function

$$f(x) = f_s(x) + \sum M_k^2 e^{-|\lambda_k| x},$$

where the sum extends over all nonzero eigenvalues, and solve the equation

$$f(x+t) + K(x,t) + \int_x^\infty K(x,\xi) f(t+\xi)\, d\xi = 0 \qquad (0 < x < t). \tag{7}$$

If the initial data satisfy the conditions of Theorem 1 or 2 (as we assume), then equation (7) has the unique solution $K(x,t)$.

In the case of the regular problem ($n = -1, 0$), the desired potential is the function

$$V_0(x) = -2\frac{d}{dx}K(x,x), \qquad x > 0.$$

If a problem with a singularity ($n \geq 1$) is being reconstructed, then the potential $V_0(x)$ and the corresponding equation

$$y'' - V_0(x)y + \lambda^2 y = 0, \qquad \int^{\infty} x^{1+\theta}|V_0(x)|\,dx < \infty \quad (\theta < \alpha), \tag{8}$$

are regarded as intermediate. The function

$$l(x,\lambda) = e^{-i\lambda x} + \int_x^{\infty} K(x,t)e^{-i\lambda t}\,dt \qquad (\operatorname{Im}\lambda \leq 0)$$

is one of the solutions of equation (8).

We find a solution $z_0(x)$ of equation (8) for $\lambda = 0$ such that:

$$z_0(0) = 0, \qquad \text{if } n \text{ is even or } n \text{ is odd and } \lambda_p = 0, \tag{9}$$

$$z_0(x) = l(x,0), \qquad \text{if } n \text{ is odd and there is no zero eigenvalue,}$$

and construct the function

$$y_k(x) = -z_0(x)\left[\int_0^x z_0^2(t)\,dt\right]^{-1}.$$

The function $y_k(x)$ satisfies the second intermediate equation

$$y'' - \left[V_k(x) + \frac{k(k+1)}{x^2}\right]y + \lambda^2 y = 0, \qquad \int^{\infty} x^{1+\theta}|V_k(x)|\,dx < \infty \quad (\theta < \alpha), \tag{10}$$

where

$$V_k(x) + \frac{k(k+1)}{x^2} = V_0(x) + 2\frac{d}{dx}\left[z_0(x)y_k(x)\right].$$

Here $k = 2$ when n is even and $k = 1$ when n is odd.

Let $z_k(x)$ be a solution of equation (10) for $\lambda = 0$, vanishing at zero. From this solution, analogously to the preceding case, we find the next intermediate equation. Continuing this process, we shall ultimately obtain the last intermediate equation

$$y'' - \left[V_{n-2}(x) + \frac{(n-2)(n-1)}{x^2}\right] y + \lambda^2 y = 0, \qquad \int^{\infty} x^{1+\theta} |V_{n-2}(x)|\, dx < \infty \quad (\theta < \alpha)$$

and its solution for $\lambda = 0$, $z_{p-2}(x)$, vanishing at zero. Setting

$$y_n(x) = \begin{cases} -z_{n-2}(x) \left[\int_0^x z_{n-2}^2(t)\, dt\right]^{-1}, & \text{in the absence of a zero eigenvalue,} \\ -z_{n-2}(x) \left[M_p^{-2} + \int_0^x z_{n-2}^2(t)\, dt\right]^{-1}, & \text{for } \lambda_p = 0, \end{cases}$$

we finally reconstruct the desired equation (1), where

$$V(x) + \frac{n(n+1)}{x^2} = V_{n-2}(x) + \frac{(n-2)(k-1)}{x^2} + 2\frac{d}{dx}\left[z_{n-2}(x) y_n(x)\right].$$

In this case the potential $V(x)$ satisfies condition (3).

Let us note that after equation (7) has been solved, all further computations reduce to simple algebraic operations and quadratures, since in solving each of the equations (8), (10), etc., one solution of this equation was already known.

5. Examples.

1) $s(\lambda) = 1$. It is easy to see that all the conditions of Theorems 1 and 2 are fulfilled, provided there are no eigenvalues. Here $f_s(x) = f(x) \equiv 0$, $K(x,t) \equiv 0$, $V_0(x) \equiv 0$. Thus, in the regular case the reconstructed equation has the form $y'' + \lambda^2 y = 0$. For $n = 0$ the functions $u(x, \lambda)$ discussed in Section 2 are $\sin \lambda x$. For $n = -1$ the boundary condition has the form $y'(0) = 0$, and $u(x, \lambda) = \cos \lambda x$. If $n = 1$, then (see (9)) $z_0(x) = 1$, which leads to the equation

$$y'' - \frac{2}{x^2} y + \lambda^2 y = 0.$$

For $n = 2$ we obtain $z_0(x) = x$ and the equation

$$y'' - \frac{6}{x^2} y + \lambda^2 y = 0.$$

2) $s(\lambda) = \dfrac{\lambda + i}{\lambda - i}$. Condition 3) of Theorem 2 is not satisfied, and hence the corresponding problem with a singularity does not exist. The conditions of Theorem 1 are satisfied; moreover, for $n = 0$, $p = 0$ (see (6)), while for $n = -1$, $p = 1$. The reconstructed equation for $n = 0$ will be

$$y'' + \frac{8}{\operatorname{ch}^2 x} y + \lambda^2 y = 0.$$

For $n = -1$, $\lambda_1^2 = -1$, $M_1^2 = 2$, we obtain the equation $y'' + \lambda^2 y = 0$. The boundary condition in this case will be $y'(0) + y(0) = 0$, and the normalized eigenfunction is $u(x, \lambda_1) = \sqrt{2}\, e^{-x}$.

3) $s(\lambda) = \dfrac{\lambda - i}{\lambda + i}$. Here $s(\lambda)$ can be a scattering function only for $n = -1$, because for $n \geq 1$ condition 3) of Theorem 2 is not satisfied, and for $n = 0$ condition 2) of Theorem 1 is not satisfied, since formula (6) gives the value (-1) for the number of eigenvalues p. For $n = -1$, $p = 0$. The corresponding equation is $y'' + \lambda^2 y = 0$, and the boundary condition is $y'(0) - y(0) = 0$.

The author expresses gratitude to Prof. V. A. Marchenko for posing the problem and supervising the work.

Kharkov State University
named after A. M. Gorky

Received
2 II 1960

REFERENCES

1. Z. S. Agranovich; V. A. Marchenko, DAN, **113**, No. 5, 951 (1957).

2. Z. S. Agranovich, V. A. Marchenko, DAN, **118**, No. 6, 1055 (1958).

Note: Figure translations are in progress. See original paper for figures.