B. B. KADOMTSEV and V. E. ROKOTYAN

1960

SovietRxiv

Abstract

Full Text

B. B. KADOMTSEV and V. E. ROKOTYAN

ON THE STABILITY OF PLASMA IN THE FIELD OF A MAGNETIC DIPOLE

(Presented by Academician M. A. Leontovich, 29 February 1960)

The field of a magnetic dipole is the simplest trap with magnetic mirrors. Under real conditions such a trap is the Earth' s magnetic field, and the existence of ion belts around the Earth ([1,2]) directly demonstrates its effectiveness. Meanwhile, it is known ([3,4]) that plasma in traps of this type is, generally speaking, unstable. It is therefore of interest to consider theoretically the stability conditions for a plasma that is in equilibrium in a dipole field.

For simplicity we shall use the hydrodynamic approximation, which in the case of an isotropic distribution of particles with respect to velocities gives a somewhat stricter stability criterion than the exact kinetic treatment ([5]).

If the surface of the dipole does not conduct current, then the most dangerous instability is the convective, or interchange, instability ([3,4]). For the field of a point dipole the corresponding stability condition ([4]) takes the form

$$-R\frac{dp}{dR} < 4\gamma p, \tag{1}$$

where p is the plasma pressure, R is the distance in the equatorial plane from the dipole to the given field line, and γ is the adiabatic exponent.

However, owing to the presence of the comparatively dense ionosphere, the Earth' s surface is rather an ideal conductor, as a result of which the tangential component of the electric field, and consequently also the displacement of the plasma near the Earth, are zero, i.e., the ends of the field lines are "frozen" into the surface. This leads to the suppression of convective instability, i.e., to additional stabilization of the plasma.

To investigate this effect we shall use the energy principle ([6]), according to which, for plasma stability, it is necessary and sufficient that the potential energy V of small oscillations be positive. Consequently, loss of plasma stability is equivalent to the vanishing of the minimum value of the potential energy

$$V = \frac{1}{8\pi}\int \left(\mathrm{rot}[\vec{\eta}\mathbf{H}]\right)^2\, dr - \frac{1}{2}\int [\vec{\eta}\mathbf{j}]\ \mathrm{rot}[\vec{\eta}\mathbf{H}]\, dr +$$

$$+\frac{1}{2}\int \vec{\eta}\nabla p\ \mathrm{div}\,\vec{\eta}\,dr + \frac{1}{2}\int \gamma p\,(\mathrm{div}\,\vec{\eta})^2\,dr, \tag{2}$$

where $\vec{\eta}$ is the displacement of the plasma from the equilibrium position, $\mathbf{H}$ is the equilibrium magnetic field, and $\mathbf{j}$ is the current density.

We shall assume that the plasma pressure is small in comparison with the pressure of the magnetic field, since only in this case can the magnetic field be regarded as dipolar:

$$H_r = \frac{2\cos\theta}{r^3}, \qquad H_\theta = \frac{\sin\theta}{r^3}, \qquad H_\varphi = 0. \tag{3}$$

Then in expression (2) the first term is much larger than the others, and this is what we shall use below.

In relation (2), all terms except the last contain only the transverse component $\vec{\eta}_\perp$, perpendicular to the magnetic field. Since the last term is small, we put $\vec{\eta} = \vec{\eta}_\perp = \{\xi\sin\theta, -2\xi\cos\theta, \alpha\}$, where ξ, α are arbitrary quantities, and later take into account the contribution from the longitudinal displacement as a certain perturbation.

Introduce the orthogonal coordinate system ψ, ζ, φ, where $\psi = 1/R = \sin^2\theta/r$ is the current function ($\mathbf{H}\nabla\psi = 0$), $\zeta = \cos\theta/r^2$ is the magnetic-field potential ($\mathbf{H} = \nabla\zeta$), and φ is the azimuth. Expressing from the equilibrium equation

$$\nabla p = [\mathbf{jH}] \tag{4}$$

the current density j through the pressure gradient and taking into account that, according to (4), $\mathbf{H}\nabla p = 0$, i.e. the pressure is a function only of ψ, we obtain

$$V = \int \left\{ \frac{1}{8\pi}\left(\mathrm{rot}[\vec{\eta}\mathbf{H}]\right)^2 - 3\frac{dp}{d\psi}\xi^2\frac{\sin^2\theta(1+\cos^2\theta)}{r^3} + \frac{1}{2}\gamma p(\mathrm{div}\,\vec{\eta})^2 \right\} \frac{d\psi\,d\zeta\,d\varphi}{H^2}. \tag{5}$$

We must find the minimum of this expression. Let η depend on the azimuth as $e^{im\varphi}$. It is not difficult to verify that, with the exception of the third component of $\mathrm{rot}[\vec{\eta}\mathbf{H}]$, the azimuthal displacement α enters everywhere in the product with m. Therefore, if we put $m\alpha = \beta$ and let $m \to \infty$, leaving β fixed, this will only decrease the potential energy.

Next, we minimize (5) with respect to β. In doing so one may disregard the last term, which is small. Substituting the value found for β in (5), we obtain

$$V = 2\pi\int\{(a\xi' + b\xi)^2 + (c+d)\xi^2\}\,d\zeta\,d\psi, \tag{6}$$

where $\xi' = d\xi/d\zeta$,

$$a = (1+3\cos^2\theta)/\sqrt{8\pi}\, r^3, \quad b = 3\cos\theta(3+\cos^2\theta)/\sqrt{8\pi}\, r(1+3\cos^2\theta),$$

$$c = -3\frac{dp}{d\psi} r^3 \sin^2\theta(1+\cos^2\theta)(1+3\cos^2\theta)^{-2}, \quad d = \gamma p(\operatorname{div}\vec{\eta}_\perp)^2/2H^2,$$

$$\operatorname{div}\vec{\eta}_\perp = 6\xi(1+\cos^2\theta)\sin\theta/r(1+3\cos^2\theta).$$

Variation of the functional (6) with respect to ξ leads to the equation $z'' - Uz = 0$, where $z = a\xi$, $U = [b^2 - (ab)' + aa'' + c + d]a^{-2}$, and thus our problem proves to be analogous to the quantum-mechanical problem of the appearance of a level with zero energy for a particle moving in the potential well U. This well turns out to be very narrow: $|U|$ decreases by half at $\zeta \sim 0.2/R$. Therefore one may use perturbation theory, regarding the well as δ-shaped. This gives us the required stability condition

$$\int_{-\infty}^{+\infty} U\, d\zeta = \int [(b-a')^2 + c + d]\frac{d\zeta}{a^2} > -\frac{2}{\zeta_0}, \tag{7}$$

where ζ_0 is the point of intersection of the field line with the conducting surface. If this surface is a sphere of radius a, then $\zeta_0 = \dfrac{\cos\theta_0}{a^2} = \dfrac{R^2}{a^2}\sqrt{1-\dfrac{a}{R}}$, where θ_0 is the coordinate of the mentioned point in the spherical coordinate system.

Let us now take into account the longitudinal displacement $\eta_\parallel$; for this purpose, in the expression for d we put

$$\operatorname{div}\vec{\eta} = \operatorname{div}\vec{\eta}_\perp + \frac{1}{H^2}\frac{\partial}{\partial\zeta}(H^2\eta_\parallel)$$

and minimize (6) with respect to $\eta_\parallel$. Then

condition (7) takes the form

$$\int [(a'-b)^2 + c]\, a^{-2}\, d\xi + \frac{1}{2}\gamma p\left(\int \frac{\operatorname{div}\vec{\eta}_\perp}{H^2}\, d\xi\right)^2 \left(\int \frac{d\xi}{H^2}\right)^{-1} > -\frac{2}{\xi_0}. \tag{8}$$

Carrying out the numerical integration, we finally obtain

$$-R\frac{dp}{dR} < \frac{H^2}{8\pi}\left[0.9 + 1.2\left(\frac{a}{R}\right)^2\left(1-\frac{a}{R}\right)^{-1/2}\right] + 4\gamma p. \tag{9}$$

In comparison with (1), an additional term has appeared here, taking into account the frozen-in character of the lines of force. For comparison with experimental data it is more convenient to write this condition in terms of the derivative of the magnetic field in the equatorial plane. Put $H = (1 - \Delta)/R^3$, where Δ characterizes the deviation of the field from the dipole field due to the presence of the current j. Taking into account that Δ is a symmetric function of $\theta - \pi/2$, we find:

$$j = \frac{1}{4\pi}(\mathrm{rot}\,\mathbf{H})_\varphi = -\frac{1}{4\pi R^3}\frac{\partial\Delta}{\partial R}.$$

Expressing, with the aid of (4), the derivative of the pressure through the current density, we write (9) in the form

$$-R\frac{\partial\Delta}{\partial R} < \frac{1}{2}\left[0.9 + 1.2\left(\frac{a}{R}\right)^2\left(1 - \frac{a}{R}\right)^{-1/2}\right] + \frac{16\pi\gamma p}{H^2}. \tag{10}$$

The Earth' s magnetic field in the region of the second belt was measured experimentally ([7]), and these data directly give Δ/R^3, i.e., the deviation of the field from the dipole field. Taking approximately $a/R = 0$, $\gamma = 5/3$, $4\pi p/H^2 = \Delta$, one can verify that condition (10) is satisfied everywhere, with the exception of a narrow interval from 21,000 to 22,000 km, where the magnetic field increases too steeply with R.

Received
4 I 1960

References

1. S. N. Vernov, A. E. Chudakov et al., DAN, **125**, 304 (1959).
2. J. Van Allen, L. Frank, *Nature*, **183**, 430 (1959).
3. M. Rosenbluth, C. Longmire, *Ann. of Phys.*, **1**, 120 (1957).
4. B. B. Kadomtsev, *Collection: Plasma Physics*, **4**, 1958, p. 380.
5. M. Kruskal, C. Oberman, *Phys. of Fluids*, **1**, 275 (1958).
6. J. Bernstein, E. Frieman, M. Kruskal, R. Kulsrud, *Proc. Roy. Soc.*, **A224**, 17 (1958).
7. S. Sh. Dolginov, N. V. Pushkov, DAN, **129**, 77 (1959).

Note: Figure translations are in progress. See original paper for figures.