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Abstract

Full Text

MATHEMATICS

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CENTERS OF NONCOMPACT SIMPLE LIE GROUPS

(Presented by Academician P. S. Aleksandrov, 16 IV 1960)

The centers of the simply connected covering groups of noncompact classical groups were found by A. S. Solodovnikov ⁽¹⁾. In the present paper the centers of all connected simply connected noncompact simple Lie groups are computed. This is done by means of a method used by E. B. Dynkin and A. L. Onishchik ⁽²⁾ for finding the centers of compact (and hence also simple complex) groups, and here extended to arbitrary simple groups. As a result, each element of the center of a group turns out to be represented by a certain vector in the Lie algebra which is carried into it under the canonical mapping.

All groups in what follows are assumed to be connected.

Let P be a compact simple Lie algebra, H its Cartan subalgebra, and $[P]$ its complex form. If Σ is the complete system of roots of P , and e_α is the root vector of $[P]$ corresponding to the root α , then we write the structural formulas in the form

$$[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha + \beta} \quad (\alpha + \beta \neq 0); \quad [e_\alpha, e_{-\alpha}] = 2\pi i d;$$

$$[h, e_\alpha] = 2\pi i (h, \alpha) e_\alpha; \quad h \in H; \quad \alpha, \beta \in \Sigma, \quad \Sigma \subset H.$$

According to F. R. Gantmacher ⁽³⁾, every involutive automorphism of the algebra P is conjugate to a transformation of the adjoint group by an automorphism $\tau = \tau_0 \exp(\bar{h})$, where \bar{h} is the matrix of the linear transformation $x \rightarrow [x, h]$, $x \in P$, $h \in H$, and τ_0 is an involutive automorphism of P carrying into itself some system $\Pi(P) \subset H$ of simple roots of the algebra P ; moreover, if τ_0 is extended to an automorphism of the whole algebra $[P]$, then $\tau_0(e_\alpha) = e_{\tau_0(\alpha)}$, $\alpha \in \Pi(P)$. The automorphism τ_0 is either an outer automorphism of the algebra P , or the identity.

Denote by P_+ the subalgebra of P belonging to the characteristic root 1 of the automorphism τ , and put $H_+ = P_+ \cap H$. Then H_+ is a Cartan subalgebra of the algebra P_+ . If P_- is the subspace of P belonging to the characteristic root -1 of the automorphism τ , then the algebra $G = P_+ + iP_-$ is a real form of the algebra

[P] ⁽⁴⁾. The real forms of simple algebras and the involutive automorphisms defining them are listed, for example, in ⁽³⁾. Denote by \mathfrak{G} some simple real group with Lie algebra G , and by \mathfrak{H}_+ its commutative subgroup generated by the subalgebra H_+ . When \mathfrak{G} is simply connected, we shall denote it by $\widetilde{\mathfrak{G}}$.

Lemma 1. *The center $\mathfrak{C}(\mathfrak{G})$ of the group \mathfrak{G} is contained in the commutative subgroup \mathfrak{H}_+ .*

Proof. We first carry out the proof for $\widetilde{\mathfrak{G}}$. Denote by \mathfrak{P}_+ the subgroup of $\widetilde{\mathfrak{G}}$ corresponding to the subalgebra P_+ , and show that $\mathfrak{C}(\widetilde{\mathfrak{G}}) \subset \mathfrak{P}_+$. In the adjoint group \mathfrak{G}^* the subgroup \mathfrak{P}_+^* , corresponding to the subalgebra P_+ , is maximal compact and there is a decomposition into a topological product $\mathfrak{G}^* = \mathfrak{P}_+^* \times \mathcal{L}$, where \mathcal{L} is a Euclidean space of the corresponding dimension ⁽⁵⁾. Since $\widetilde{\mathfrak{G}}$ is simple

the natural homomorphism $p : \widetilde{\mathcal{G}} \rightarrow \mathcal{G}^*$ is a covering map. But then the group $p^{-1}(\mathfrak{P}_+^*)$, being, by virtue of the relation $\mathcal{G}^* = \mathfrak{P}_+^* \times \mathcal{L}$, a connected covering group of \mathfrak{P}_+^* , is locally isomorphic to \mathfrak{P}_+^* ; whence it follows that $p^{-1}(\mathfrak{P}_+^*) = \mathfrak{P}_+$. $\mathfrak{C}(\widetilde{\mathcal{G}}) = p^{-1}(e)$, where e is the identity of \mathfrak{P}_+^* ; consequently, $\mathfrak{C}(\widetilde{\mathcal{G}}) \subset \mathfrak{P}_+$.

The compact algebra P_+ admits a decomposition into a direct sum $P_+ = P_1 + V$, where P_1 is compact semisimple, and V is a commutative algebra (whose dimension is zero or one). Then $H_+ = H_1 + V$, where H_1 is a Cartan subalgebra of P_1 . Correspondingly, in view of the simple connectedness of $\widetilde{\mathcal{G}}$, and hence also of \mathfrak{P}_+ , $\mathfrak{P}_+ = \mathfrak{P}_1 \cdot \mathcal{V}$, $\mathcal{H}_+ = \mathcal{H}_1 \cdot \mathcal{V}$, where \mathfrak{P}_1 is a semisimple compact subgroup of $\widetilde{\mathcal{G}}$ with Lie algebra P_1 , \mathcal{H}_1 is its maximal torus with Lie algebra H_1 , and \mathcal{V} is a simply connected commutative subgroup of $\widetilde{\mathcal{G}}$ with Lie algebra V . From what was proved above it follows that $\mathfrak{C}(\widetilde{\mathcal{G}}) \subset \mathfrak{C}(\mathfrak{P}_+)$; but, since the center of a compact group is contained in its maximal torus, $\mathfrak{C}(\mathfrak{P}_+) = \mathfrak{C}(\mathfrak{P}_1) \cdot \mathcal{V} \subset \mathcal{H}_1 \cdot \mathcal{V} = \mathcal{H}_+$.

The extension of the lemma to the case of a non-simply connected \mathcal{G} is carried out with the aid of the factorization of the group $\widetilde{\mathcal{G}}$.

For an arbitrary semisimple compact algebra R , denote by $\Gamma_0(R)$ the integral lattice in the Cartan subalgebra of R whose basis consists of the vectors $\alpha' = \frac{2\alpha}{(\alpha, \alpha)}$, $\alpha \in \Pi(R)$, and by $\Gamma_1(R)$ the integral lattice whose basis is biorthogonal to the system $\Pi(R)$. If \mathcal{R} is a simply connected group with Lie algebra R , then $\Gamma_0(R)$ is the full inverse image of the identity in the Cartan subalgebra of the algebra R under the canonical mapping R into \mathcal{R} ⁽²⁾.

Introduce P_0 into consideration—the subalgebra of P belonging to the characteristic root 1 of the automorphism τ_0 . P_0 is always semisimple (and, evidently, compact), and H_+ is a Cartan subalgebra for it.

Lemma 2. *The inverse image of the center of the group \mathcal{G} in H_+ under the canonical mapping $c : G \rightarrow \mathcal{G}$ is the lattice $\Gamma_1(P) \cap H_+$, and it coincides with*

the lattice $\Gamma_1(P_0)$.

Proof. The canonical mapping induces a homomorphism of the additive vector group H_+ onto the group \mathcal{H}_+ . An element $h \in H_+$ belongs to $c^{-1}(\mathcal{C}(\mathcal{G}))$ if and only if the linear transformation $pc(h)$, acting on G , is identical (p is the adjoint representation of the group \mathcal{G}). The transformation $pc(h)$ extends uniquely to a linear transformation on $[P]$. It is identical if and only if $h \in \Gamma_1(P)$ (2). The second assertion of the lemma follows from the fact that the projection of the system $\Pi(P)$ onto H_+ is $\Pi(P_0)$.

Theorem. The center of the simply connected group $\tilde{\mathcal{G}}$ is isomorphic to the factor group $\Gamma_1(P_0)/\Gamma_0(P_1)$, and the isomorphism is generated by the canonical mapping $c : G \rightarrow \tilde{\mathcal{G}}$.

Proof. Let us find the inverse image of the identity e in H_+ under the canonical mapping, i.e. $c^{-1}(e) \cap H_+$. In view of the simple connectedness of $\tilde{\mathcal{G}}$, and consequently also of \mathcal{V} , the homomorphism $V \rightarrow \mathcal{V}$ generated by the canonical mapping is an isomorphism. Therefore $c^{-1}(e) \cap H_+ \subset H_1$, i.e. $c^{-1}(e) \cap H_+$ coincides with the inverse image of the identity in H_1 under the canonical mapping of the algebra P_1 into the group \mathcal{P}_1 . Since H_1 is a Cartan subalgebra of P_1 , and \mathcal{P}_1 is semisimple compact and simply connected (in view of the simple connectedness of $\tilde{\mathcal{G}}$), we have $c^{-1}(e) \cap H_+ = \Gamma_0(P_1)$. The assertion of the theorem now follows from Lemma 2.

Below are given the centers $\mathcal{C}(\tilde{\mathcal{G}})$ of simply connected noncompact simple real Lie groups, computed as complete sets of representatives of the cosets $\Gamma_1(P_0)$ modulo $\Gamma_0(P_1)$ for each Lie algebra G . $Z_m(z)$ denotes the additive cyclic group of order m with generator

z , $Z(z)$ —the infinite cyclic group. The numbering of the simple roots is the same as in (2). The real forms of exceptional algebras are specified by the signatures δ of their Cartan metric.

Real forms A_n ($n > 1$)

1. $G = A_n^l$ —the algebra of matrices of order $n + 1$ with trace 0, preserving the invariant Hermitian form

$$-\sum_1^l x_k \bar{y}_k + \sum_{l+1}^{n+1} x_k \bar{y}_k, \quad l = 1, \dots, \left[\frac{n+1}{2} \right].$$

Introduce the notation:

$$u_1 = \frac{1}{n+1} \sum_1^n k \alpha_k, \quad u_2 = \frac{1}{n+1} \sum_1^n (n-k+1) \alpha_k,$$

$d = \gcd(l, n-l+1)$, $z_1 = \frac{l}{d}u_2 - \frac{n-l+1}{d}u_1$, $z_2 = M_1u_1 + M_2u_2$, where M_1, M_2 are integers such that $M_1l + M_2(n-l+1) = d$. Then the center has the form $Z_d(z_1) + Z(z_2)$.

2. $G = I_n$ —the algebra of real matrices of order $n+1$ with trace 0. Introduce the notation

$$z = \frac{1}{2} \sum_0^{[n/2]} \alpha_{2k+1}, \quad z_1 = \alpha_{\frac{n+1}{2}}, \quad z_2 = \alpha_{\frac{n}{2}} + \alpha_{\frac{n}{2}+1}.$$

Then the center has the form $Z_4(z)$, if n and $(n+1)/2$ are odd; $Z_2(z) + Z_2(z_1)$, if n is odd and $(n+1)/2$ is even; $Z_2(z_2)$, if n is even.

3. $G = J_n$ (n odd)—the algebra of quaternionic matrices of order $(n+1)/2$, considered up to real positive factors. The center has the form $Z_2(z)$, where

$$z = \frac{1}{2} \sum_0^{[n/2]} \alpha_{2k+1}.$$

Real forms B_n

1. $G = B_n^{2l}$ —the algebra of real matrices of order $2n+1$, preserving the invariant quadratic form

$$-\sum_1^{2l} x_k^2 + \sum_{2l+1}^{2n+1} x_k^2, \quad l = 1, \dots, n.$$

The center has the form $Z(z_1) + Z_2(z_2)$ for $l = 1$, and $Z_2(z_1) + Z_2(z_2)$ for $l > 1$, where $z_1 = \alpha'_l$, $z_2 = \frac{1}{2}\alpha'_n$.

Real forms C_n

Denote

$$z = \frac{1}{2} \sum_0^{[(n-1)/2]} \alpha'_{2k+1}, \quad z_1 = \alpha'_n.$$

1. $G = C_n^{2l}$ —the algebra of matrices of order $2n$, preserving the invariant skew-symmetric bilinear form

$$\sum_1^n (x_{2k-1}y_{2k} - x_{2k}y_{2k-1})$$

and the Hermitian form

$$-\sum_1^{2l} x_k \bar{y}_k + \sum_{2l+1}^{2n} x_k \bar{y}_k.$$

The center has the form $Z_2(z)$.

2. $G = IC_n$ —the algebra of real matrices of order $2n$, preserving the invariant skew-symmetric bilinear form

$$\sum_1^n (x_{2k-1} y_{2k} - x_{2k} y_{2k-1}).$$

The center has the form $Z(z)$, if n is odd, and $Z_2(z) + Z(z_1)$, if n is even.

Real forms D_n

Let

$$z = \frac{1}{2}(\alpha_{n-1} + \alpha_n), \quad z_1 = \frac{1}{2} \sum_1^{[(n-1)/2]} \alpha_{2k-1} + \frac{n}{4}(\alpha_{n-1} + \alpha_n) - \frac{1}{2}\alpha_n.$$

$$z_2 = \alpha_l + \frac{1}{2}(\alpha_{n-1} + \alpha_n), \quad z_3 = \frac{1}{2} \sum_1^{[(n-1)/2]} \alpha_{2k-1} + \frac{n}{4}(\alpha_{n-1} + \alpha_n) - \frac{1}{2}\alpha_{n-1}.$$

$$z_4 = \alpha_l.$$

1. $G = D_n^{2l}$ —the algebra of real matrices of order $2n$ leaving invariant the quadratic form

$$-\sum_1^{2l} x_k^2 + \sum_{2l+1}^{2n} x_k^2, \quad l = 1, \dots, \left[\frac{n}{2}\right].$$

The center has the form $Z_2(z) + Z(z_1)$, if $l = 1$; $Z_4(z_1) + Z_2(z_4)$, if $l > 1$, n is odd; $Z_4(z_1) + Z_2(z_2)$, if $l > 1$, n is even, l is odd; $Z_2(z_1) + Z_2(z_2) + Z_2(z_3)$, if $l > 1$, n and l are even.

2. $G = JD_n$ —the algebra of matrices of order $2n$ leaving invariant the quadratic form

$$\sum_1^n x_{2k-1} x_{2k}$$

and the Hermitian form

$$\sum (x_{2k-1}\bar{x}_{2k-1} - x_{2k}\bar{x}_{2k}).$$

The center has the form $Z(z_1 - sz)$, if $n = 2s + 1$; $Z_2(z_1 - sz) + Z(z_3 - sz)$, if $n = 4s + 2$; $Z(z_1 - 2sz) + Z_2(z_3 - 2sz)$, if $n = 4s^*$.

3. $G = D_n^{2l+1}$ —the algebra of real matrices of order $2n$ leaving invariant the quadratic form

$$-\sum_1^{2l+1} x_k^2 + \sum_{2l+1}^{2n} x_k^2, \quad l = 0, 1, \dots, \left[\frac{n}{2} \right].$$

The center has the form $Z_2(z) + Z_2(z_4)$.

Real form G_2

1. $\delta = 2$. The center has the form $Z_2(\alpha'_2)$.

Real forms F_4

1. $\delta = -52$. The center is trivial.
2. $\delta = 4$. The center has the form $Z_2(\alpha'_1)$.

Real forms E_6

Put $z = \frac{1}{3}(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5)$, $z_1 = \alpha_6$.

1. $\delta = -14$. The center has the form $Z(z)$.
2. $\delta = 2$. The center has the form $Z_6(z)$.
3. $\delta = -26$. The center is trivial.
4. $\delta = 6$. The center has the form $Z(z_1)$.

Real forms E_7

Put $z = \frac{1}{2}(\alpha_4 + \alpha_6 + \alpha_7)$, $z_1 = \alpha_5$.

1. $\delta = -5$. The center has the form $Z_2(z) + Z_2(z_1)$.
2. $\delta = 7$. The center has the form $Z_4(z)$.
3. $\delta = -25$. The center has the form $Z(z)$.

Real forms E_8

1. $\delta = -24$. The center has the form $Z_2(\alpha_1)$.
2. $\delta = 8$. The center has the form $Z_2(\alpha_7)$.

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CITED LITERATURE

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* What is stated in the corresponding item in (1) is valid only for odd n .

Note: Figure translations are in progress. See original paper for figures.

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