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Abstract

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MATHEMATICS

G. I. ESKIN

A SUFFICIENT CONDITION FOR THE SOLVABILITY OF THE MULTIDIMENSIONAL MOMENT PROBLEM

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Let Φ_1 and Φ_2 be linear spaces with involution, and let A and B be linear operators in Φ_1 and Φ_2, respectively, real with respect to the involution, i.e. $A\varphi^* = (A\varphi)^*$ for every $\varphi \in \Phi_1$ and $B\psi^* = (B\psi)^*$ for every $\psi \in \Phi_2$. Let $\Phi = \Phi_1 \otimes \Phi_2$ be the tensor product of Φ_1 and Φ_2, i.e. the set of elements x of the form $\sum_{k=1}^{n} \varphi_k \otimes \psi_k$, where $\varphi_k \in \Phi_1,\ \psi_k \in \Phi_2$. The operators A and B and the involution are defined in Φ in the natural way. Suppose that a scalar product (x, y), real with respect to the involution, is given in Φ, i.e. $(x^*, y^*) = \overline{(x, y)}$, and that A and B are symmetric operators in this scalar product. As a consequence of the reality of A, and also of B, they have equal deficiency indices. Denote by H the completion of Φ with respect to the scalar product (x, y). For fixed $\psi_0 \in \Phi_2$, obviously, $(\varphi_1 \otimes \psi_0, \varphi_2 \otimes \psi_0)$ is a scalar product in Φ_1.

Theorem 1. *Let $\bar{A}_{\psi_0}$, the closure of A in the Hilbert space H_{ψ_0} obtained by completing Φ_1 with respect to the scalar product $(\varphi_1 \otimes \psi_0, \varphi_2 \otimes \psi_0)$, be a self-adjoint operator for every fixed $\psi_0 \in \Phi_2$.*

Then $\bar{A}$, the closure of A in H, is a self-adjoint operator, and in H there exists a self-adjoint extension $\widetilde{B}$ of the operator $\bar{B}$ such that $\bar{A}$ and $\widetilde{B}$ commute in H, i.e. their spectral families commute.

The case when the closures of A and B in H are self-adjoint operators was considered by A. G. Kostyuchenko and B. S. Mityagin in ([1]). The case when only A is self-adjoint was considered by R. S. Ismagilov under the assumption that Φ is a Carleman space ([2]). Theorem 1 generalizes R. S. Ismagilov' s result, which makes it possible to obtain certain new results for the multidimensional moment problem and the representation of positive-definite functionals.

Proof of Theorem 1. The operators A and B, obviously, commute on Φ. Therefore, for any non-real λ the equality

$$R_\lambda B x = B R_\lambda x \quad \text{for } x \in \Phi_\lambda = (A - \lambda E)\Phi_1 \otimes \Phi_2, \tag{1}$$

holds, where R_λ is the resolvent of the operator $\bar{A}$. Denote by B_λ the operator B considered on the linear manifold Φ_λ. Equality (1) holds for any $x \in D_{\bar{B}_\lambda}$, where $\bar{B}_\lambda$ is the closure of B_λ in H, since R_λ is a bounded operator. We shall show that $\bar{B}_\lambda \supset B$, i.e. $D_{\bar{B}_\lambda} \supset \Phi$ and for $x \in \Phi$ one has $\bar{B}_\lambda x = Bx$.

Indeed, let φ_0 be any element of Φ_1 and ψ_0 any element of Φ_2. From the condition of the theorem it follows that the set $(A - \lambda E)\Phi_1$ is dense in H_{ψ_1}, where $\psi_1 = (B - iE)\psi_0 \in \Phi_2$, so that there exists a sequence $\varphi_n^{(1)} = (A-$

$-\lambda E)\varphi_n$, where $\varphi_n \in \Phi_1$, converging to φ_0 in the norm H_{ψ_1}. But, since B is symmetric, $\|(B - iE)x\|^2 = \|Bx\|^2 + \|x\|^2$, so that convergence in H_{ψ_1} means that simultaneously

$$(A - \lambda E)\varphi_n \otimes \varphi_0 \to \varphi_0 \otimes \psi_0, \qquad (A - \lambda E)\varphi_n \otimes B\psi_0 \to \varphi_0 \otimes B\psi_0$$

in the norm H. Consequently, $\varphi_0 \otimes \psi_0 \in D_{\overline{B}_\lambda}$ and, hence, $\Phi \subset D_{\overline{B}_\lambda}$. On the other hand, since $B_\lambda \subset B$, we have $\overline{B}_\lambda \subset \overline{B}$. Thus, $\overline{B}_\lambda = \overline{B}$. Hence, for any $x \in \Phi$ we have $\overline{B} R_\lambda x = R_\lambda \overline{B} x$. It follows that for any $y \in D_{B^*}$,

$$(\overline{B} R_\lambda x, y) = (R_\lambda x, B^* y) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \xi}\, d(E_\xi x, B^* y).$$

Next,

$$(R_\lambda \overline{B} x, y) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \xi}\, d(E_\xi \overline{B} x, y).$$

Comparing the last two equalities, from the uniqueness of the Stieltjes transform we have, for any ξ, that $(E_\xi \overline{B} x, y) = (E_\xi x, B^* y)$. The equality

$$(\overline{B} x, E_\xi y) = (x, E_\xi B^* y)$$

holds for all $x \in \Phi$. It follows that if $y \in D_{B^*}$, then also $E_\xi y \in D_{B^*}$, and

$$B^* E_\xi y = E_\xi B^* y, \tag{2}$$

i.e., the operator B^* commutes with E_ξ.

Further, $D_{B^*} = D_{\overline{B}} \dotplus M_+ \dotplus M_-$, where $B^* M_+ = iM_+$ and $B^* M_- = -iM_-$. From (2) it follows that $E_\xi M_\pm \subset M_\pm$ and $E_\xi D_{\overline{B}} \subset D_{\overline{B}}$. In consequence of the reality of A and B, the operators B^* and E_ξ will also be real. Therefore $M_- = M_+^*$, i.e., if $x \in M_+$, then $x^* \in M_-$. Since every self-adjoint extension of the operator B is a part of B^*, in order that some extension $\widetilde{B}$ commute with E_ξ, it is necessary that from $y \in D_{\widetilde{B}}$ it should follow that $E_\xi y \in D_{\widetilde{B}}$. Let

$$D_{\widetilde{B}} = D_{\overline{B}} \dotplus M_+ \dotplus V M_+,$$

where V is an isometric operator mapping M_+ onto M_-. Then

$$E_\xi D_{\widetilde{B}} = E_\xi D_{\overline{B}} \dotplus E_\xi M_+ \dotplus E_\xi V M_+,$$

and, in order that $E_\xi D_{\widetilde{B}} \subset D_{\widetilde{B}}$, it is necessary that $E_\xi V M_+ = V E_\xi M_+$. Let x_0 be an arbitrary element of M_+. Denote by $M_+^{(1)}$ the closure of the linear span of the vectors $E_\Delta x_0$, where Δ is an arbitrary interval. $M_+^{(1)} \subset M_+$. Let $M_-^{(1)} = M_+^{(1)*} \subset M_-$. Define the operator $V^{(1)}$ from $M_+^{(1)}$ into $M_-^{(1)}$ as follows:

$$V^{(1)} E_\Delta x_0 = E_\Delta x_0^*,$$

and on the remaining vectors by linearity and continuity. $V^{(1)}$ maps $M_+^{(1)}$ isometrically onto $M_-^{(1)}$, since the operator E_ξ and the scalar product (x, y) are real with respect to the involution. It is obvious that

$$E_\xi V^{(1)} M_+^{(1)} = V^{(1)} E_\xi M_+^{(1)}$$

for any ξ. If the orthogonal complement to $M_+^{(1)}$ in M_+ is not empty, then choose there an arbitrary element y_0 and do the same, etc., until M_+ is exhausted.

Thus one can construct an isometric operator V such that $E_\xi V M_+ = V E_\xi M_+$. The corresponding extension $\widetilde{B}$ will commute with E_ξ: $\widetilde{B} E_\xi x = E_\xi \widetilde{B} x$ for all $x \in D_{\widetilde{B}}$. Such an extension $\widetilde{B}$ is not unique, if $\overline{B}$ is not a self-adjoint operator. Let R_μ be the resolvent of $\widetilde{B}$. Since

$$(\widetilde{B} - \mu E) E_\xi x = E_\xi (\widetilde{B} - \mu E) x,$$

then, applying R_μ to both sides of the equality, we obtain

$$E_\xi R_\mu y = R_\mu E_\xi y$$

for all $y \in H$. Using again the spectral representation of the resolvent and the uniqueness of the Stieltjes transform, we shall have $E_\xi F_\eta = F_\eta E_\xi$, where F_η is the spectral family of $\widetilde{B}$. The theorem is proved.

Let now Φ_1 and Φ_2 be nuclear algebras, and let the scalar product be given by a continuous positive-definite functional $T(x)$, i.e. one such that $T(x \cdot x^*) \geq 0$. Then, assuming the conditions fulfilled

of Theorem 1, from the general scheme in (1) we obtain a representation for $T(x)$:

$$T(x) = \int X_\lambda(x)\, d\sigma(\lambda), \tag{3}$$

where X_λ are the common eigenfunctionals of the operators A' and B', adjoint to A and B in Φ'. The measure $\sigma(\lambda)$, generally speaking, is not unique.

Let us give some applications of Theorem 1.

1. Let c_{mn} be a double sequence of moments, i.e., for any sequence $\{\xi_{mn}\}$

$$\sum_{m,n} c_{m_1+m_2,\,n_1+n_2}\xi_{m_1,n_1}\bar{\xi}_{m_2,n_2} \geqslant 0. \tag{4}$$

The question is posed of when the two-dimensional moment problem is solvable (see ([3])), i.e., when there exists a nonnegative measure $\sigma(\lambda,\mu)$ such that

$$c_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda^m \mu^n \, d\sigma(\lambda,\mu).$$

Theorem 2. *The two-dimensional moment problem is solvable if, for any fixed n_0, the one-dimensional moment problem $a_m = c_{m,2n_0} + c_{m,2(n_0+1)}$ is determinate; moreover, the measure $\sigma(\lambda,\mu)$, generally speaking, is not unique.*

Theorem 2 follows from Theorem 1 if we assume that Φ_1 and Φ_2 are spaces of sequences in which only a finite number of terms are nonzero, and A and B are shift operators, i.e., $A\{\xi_m\} = \{\xi_{m-1}\}$ and $B\{\eta_n\} = \{\eta_{n-1}\}$; the involution is simply passage to the complex-conjugate sequence, and the scalar product is defined by means of (4).

2. Let $f(x,y)$ be a continuous function positive-definite in the sense of Bochner in the rectangle $P = [-a,a;\,-b,b]$, i.e.,

$$\int_P f(x_1-x_2, y_1-y_2)\varphi(x_1,y_1)\overline{\varphi(x_2,y_2)}\, dx_1dx_2dy_1dy_2 \geqslant 0 \tag{5}$$

for all finite infinitely differentiable functions from P. In this case $\Phi_1 = K(-a,a)$, $\Phi_2 = K(-b,b)$. The operators A and B are respectively $i\,d/dx$ and $i\,d/dy$. The involution is given by the equality $[\varphi(x,y)]^* = \overline{\varphi(-x,-y)}$, and the scalar product is determined by (5).

The inequality holds

$$\int_P f(x_1-x_2, y_1-y_2)\varphi(x_1)\overline{\varphi(x_2)}\psi(y_1)\overline{\psi(y_2)}\, dP \leqslant$$

$$\leqslant \left(\int_{-b}^{b} |\psi(y)|\, dy\right)^2 \int_{-a}^{a}\int_{-a}^{a} f(x_1-x_2, 0)\varphi(x_1)\overline{\varphi(x_2)}\, dx. \tag{6}$$

The function $f(x,0)$ is obviously a positive-definite function. From (6) it follows that convergence in the scalar product generated by $f(x,0)$ implies convergence in the scalar product (5) for any fixed $\psi(y)$, which is stronger than the requirement of Theorem 1. Therefore the following theorem holds:

Theorem 3. *If $f(x,0)$ admits a unique extension to the entire X-axis as a positive-definite function, then $f(x,y)$ has the representation*

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda x+\mu y)}\,d\sigma(\lambda,\mu),$$

where $\sigma(\lambda,\mu)$ is a nonnegative measure, generally speaking not unique, so that $f(x,y)$ admits an extension to the whole plane as a positive-definite function.

Devinatz in (4) proved an analogous theorem under the assumption that $\hat{f}(x,0)$ on the interval $(-a+\varepsilon, a-\varepsilon)$ admits a unique extension to the whole axis as a positive-definite function.

3. Theorem 1 is easily generalized to the case when $\Phi=\Phi_1\otimes\cdots\otimes\Phi_k$, the operators A_i $(1\le i\le k)$ act in Φ_i, and the closure $\overline{A_i}$ $(1\le i\le k-1)$ is a self-adjoint operator in the Hilbert space obtained by completing Φ_i with respect to the scalar product

$$(\varphi,\varphi)_i=(\psi_{10}\otimes\cdots\otimes\psi_{i-1,0}\otimes\varphi\otimes\psi_{i+1,0}\otimes\cdots\otimes\psi_{k0},$$

$$\psi_{10}\otimes\cdots\otimes\psi_{i-1,0}\otimes\varphi\otimes\psi_{i+1,0}\otimes\cdots\otimes\psi_{k0})$$

for arbitrary $\psi_{j0}\in\Phi_j$ $(j\ne i)$. Then there exists a self-adjoint extension $\widetilde{A}_k$ of the operator A_k such that all $\overline{A_i}$ $(1\le i\le k-1)$ commute with $\widetilde{A}_k$. The fact that the $\overline{A_i}$ commute with one another was proved in (1).

Theorem 2 takes the following form in the multidimensional case: if the one-dimensional moment problem

$$a_{n_i}=c_{2n_{10},\dots,2n_{i-1},0,n_i,2n_{i+1},0,\dots,2n_{k0}}+c_{2n_{10},\dots,2n_{i-1},0,n_i,2n_{i+1},0,\dots,2(n_{k0}+1)}$$

is uniquely solvable for $1\le i\le k-1$ and for arbitrary n_{j0} $(j\ne i)$, then the multidimensional moment problem

$$c_{n_1,\dots,n_k}=\int\lambda_1^{n_1}\cdots\lambda_k^{n_k}\,d\sigma(\lambda_1,\dots,\lambda_k)$$

is solvable and, generally speaking, not uniquely.

Similarly, if $f(x_1,\dots,x_k)$ is a positive-definite function in a k-dimensional parallelepiped, then if $f(x_1,0,\dots,0)$, $f(0,x_2,\dots,0),\dots,f(0,\dots,x_{k-1},0)$ admit unique extensions to the whole axis as positive-definite functions, then $f(x_1,\dots,x_k)$ also admits an extension to the whole k-dimensional space as a positive-definite function.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.