MATHEMATICS

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Abstract

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MATHEMATICS

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TRANSIENT PHENOMENA IN BRANCHING PROCESSES WITH n TYPES OF PARTICLES

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§ 1. Consider the branching random process defined in ([1]), with n types of particles, homogeneous in time. Let, during a time interval $\Delta t \to 0$, a particle of type T_k turn into a set of particles $\omega = (\omega_1, \dots, \omega_n)$ of types $T_1, \dots, T_n$ with probabilities $\delta_k^\omega + p_k^\omega \Delta t + o(\Delta t)$, where $\delta_k^\omega = 1$ if $\omega_k = 1$, $\omega_i = 0$ $(i \neq k)$; $\delta_k^\omega = 0$ in all other cases. The densities $p_k^\omega < 0$ if $\delta_k^\omega = 1$; $p_k^\omega \geq 0$ if $\delta_k^\omega = 0$; $\sum_\omega p_k^\omega = 0$. Denote by $\mu_{kj}(t)$ the number of particles of type T_j obtained from one particle of type T_k during time t. The generating functions $F_k(t, x_1, \dots, x_n)$ of the probabilities $P_k^\omega(t) = \mathbf{P}\{\mu_{kj}(t) = \omega_j,\ j = 1, \dots, n\}$ satisfy the system of equations ([1])

$$\frac{dF_k}{dt} = f_k(F_1, \dots, F_n), \qquad k = 1, \dots, n \tag{1}$$

(where $f_k(x_1, \dots, x_n) = \sum_\omega p_k^\omega x_1^{\omega_1} \cdots x_n^{\omega_n}$) and the boundary conditions

$$F_k(0, x_1, \dots, x_n) = x_k \qquad (k = 1, \dots, n).$$

Denote the factorial moments of the functions $f_k(x_1, \dots, x_n)$ by

$$a_{ij} = \partial f_i / \partial x_j \big|_{x=1}, \quad b_{ij}^{(k)} = \partial^2 f_k / \partial x_i \partial x_j \big|_{x=1}, \quad c_{i,j,l}^{(k)} = \partial^3 f_k / \partial x_i \partial x_j \partial x_l \big|_{x=1}.$$

For processes with an indecomposable matrix $a = \|a_{ij}\|$, as $t \to \infty$,

$$M\mu_{kj}(t) \sim u_k v_j e^{\lambda t} \qquad (k, j = 1, \dots, n),$$

where λ is the characteristic number with largest real part of the matrix a; $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ are eigenvectors of the number λ for the matrix a and its transpose, respectively, with

$$\sum_{k=1}^n v_k^2 = 1, \qquad \sum_{k=1}^n u_k v_k = 1,$$

$$u_k > 0, \quad v_k > 0 \qquad (k = 1, \dots, n).$$

Put

$$b = \sum_{i,j,k=1}^{n} b_{ij}^{(k)} v_k u_i u_j.$$

For processes with an indecomposable matrix a and $0 < b < \infty$, the probabilities of continuation of the process (begun with one particle of type T_i)

$$Q_i(t) = \mathbf{P}\left\{\sum_{j=1}^{n} \mu_{ij}(t) > 0\right\}$$

as $t \to \infty$ satisfy the relations ([1,2,6])

$$Q_i(t) \sim \begin{cases} K_i e^{\lambda t}, & \text{if } \lambda < 0, \\ 2u_i/bt, & \text{if } \lambda = 0, \\ 1 - P_i, & \text{if } \lambda > 0. \end{cases}$$

where $K_i > 0$ are constants, P_i satisfy the equations $f_k(P_1, \dots, P_n) = 0$ $(k = 1, \dots, n)$, and the conditional distribution laws ([2],[3],[6])

$$S_k(t, \lambda, y) = \mathbf{P}\left\{\frac{\mu_{kj}(t)}{M\{\mu_{kj}(t) \mid \mu_k(t) > 0\}} < y_j,\ j = 1, \dots, n \mid \mu_k(t) > 0\right\} \tag{2}$$

(where $y = (y_1, \dots, y_n)$ and $\mu_k(t) = \sum_{j=1}^{n} \mu_{kj}(t)$) converge weakly to the distributions $S_k(y)$.

For $\lambda < 0,\ \lambda > 0$, the distributions $S_k(y)$ depend essentially on the form of the functions $f_k(x)$ $(k = 1, \dots, n)$; moreover, in the first case the distributions are discrete, and in the second continuous. For $\lambda = 0$, the distributions $S_k(y)$ are exponential, with characteristic functions

$$\varphi_k(\tau_1, \dots, \tau_n) = \frac{1}{1 - i(\tau_1 + \dots + \tau_n)}. \tag{3}$$

Thus, a branching process with $\lambda = 0$ separates two mutually distinct types of branching processes. We shall call transition phenomena those phenomena that arise as $\lambda \to 0$.

In the present note we shall give limit theorems for $t \to \infty,\ \lambda \to 0$, analogous to the theorems in ([4]), obtained for processes with one particle type. The results stated above for processes with $\lambda = 0$ can be obtained as consequences of Theorems 1 and 2.

§ 2. Let $\mathfrak{A}$ denote a closed compact set (in the sense of convergence by elements) of indecomposable matrices of the form a with λ close to 0. We shall consider branching processes from the class $K(\mathfrak{A}, \delta, B, c)$. A process belongs to $K(\{f_k\} \in K)$ if $a \in \mathfrak{A},\ 0 < \delta < b < B < \infty,\ c_{ijl}^{(k)} \le c < \infty\ (c \ge 0)$. Define the function

$$k(t, x, \lambda) = \frac{e^{\lambda t} \sum_{j=1}^n v_j(1 - x_j)}{1 + \dfrac{b}{2}\, g(\lambda, t) \sum_{j=1}^n v_j(1 - x_j)},$$

where $g(\lambda, t) = t$ for $\lambda = 0$ and $g(\lambda, t) = \dfrac{e^{\lambda t} - 1}{\lambda}$ for $\lambda \ne 0$.

Theorem 1.

$$1 - F_j(t, x) = u_j k(t, x, \lambda)[1 + \eta_j(t, x, \lambda)],$$

where $\eta_j(t, x, \lambda) \to 0$ as $t \to \infty,\ \lambda \to 0$, uniformly in $\{f_k\} \in K,\ 0 \le x_k \le 1$ $(k = 1, \dots, n)$.

Corollary. As $t \to \infty,\ \lambda \to 0$, uniformly in $\{f_k\} \in K$,

$$Q_j(t) \sim \frac{2u_j e^{\lambda t}}{bg(\lambda, t)}.$$

Theorem 2. As $t \to \infty,\ \lambda \to 0$, uniformly in $\{f_k\} \in K$,

$$\max_y |S_k(t, \lambda, x) - S(y)| \to 0$$

(where $S(y)$ is defined by (3)).

We shall briefly outline the proofs of the formulated theorems.

Proof of Theorem 1. First we establish that

$$Q(t) = \sum_{k=1}^n Q_k(t) v_k \to 0$$

as $t \to \infty,\ \lambda \to 0$, uniformly in $\{f_k\} \in K$. Next put

$$r_k(t, x) = \sum_{j=1}^n \pi_{kj} R_j(t, x),$$

where $k = 1, \dots, n,\ R_j(t, x) = 1 - F_j(t, x)$. Choose the matrix $\Pi = \|\pi_{kj}\|$ so that: 1) the matrix $\Pi a \Pi^{-1}$ has Jordanew form; 2) the first row of P was proportional to v; 3) the ratios $r_k(t,x)/r_1(t,x)$ $(k \neq 1)$ were sufficiently small uniformly in t, $0 \leq x_j \leq 1$ $(j = 1, \dots, n)$, $\{f_k\} \in K$ (this is possible, since

$$|r_k| \leq \sum_{j=1}^{n} |\pi_{kj}| R_j(t,x) \leq \frac{\max_j |\pi_{kj}|}{\min_j v_j} \sum_{j=1}^{n} v_j R_j(t,x),$$

and $\min_j v_j > \delta_1 > 0$ for any $a \in \mathfrak{A}$).

To prove the theorem it is enough to show that, as $t \to \infty$, $\lambda \to 0$, uniformly for $0 \leq x_k \leq 1$ $(k = 1, \dots, n)$, $\{f_k\} \in K$,

$$\frac{r_k(t,x)}{r_1(t,x)} \longrightarrow 0 \qquad (k \neq 1); \tag{4}$$

$$r_1(t,x) \sim k(t,x,\lambda), \tag{5}$$

since it follows from (4) that $R_j(t,x) \sim u_j r_1(t,x)$.

The proof of (5) is carried out analogously to the proof of theorem 1 in ([4]). Relations (4) can be proved with the aid of the inequality

$$d\rho^2/dt < -\delta_0\rho^2 + CQ(t) \qquad \left(\delta_0 > 0,\ C > 0 \text{ are constants, } \rho^2 = \sum_{k=2}^{n} |r_k|^2 \Big/ \sum_{k=2}^{n} |r_k|^2\right),$$

which can be obtained from (1) by applying transformations analogous to the transformations in the proof of theorem 2, p. 121 in ([5]), if one takes into account that $r_k(t,x)/r_1(t,x)$ $(k \neq 1)$ are uniformly small.

Proof of theorem 2. If theorem 2 is false, then there exist $\varepsilon > 0$ and a sequence $S_k(t^{(m)}, \lambda^{(m)}, y)$ such that

$$\max_y |S_k(t^{(m)}, \lambda^{(m)}, y) - S(y)| > \varepsilon$$

and $t^{(m)} \to \infty$, $\lambda^{(m)} \to 0$ as $m \to \infty$. By theorem 1 it is not difficult to compute that the Laplace transforms of the distributions $S^k(t^{(m)}, \lambda^{(m)}, y)$ as $m \to \infty$ converge to

$$\frac{1}{1 + s_1 + \dots + s_n}$$

$(s_j \geq 0,\ j = 1, \dots, n)$. Since $S(y)$ is continuous, it follows from this that

$$\max_y |S_k(t^{(m)}, \lambda^{(m)}, y) - S(y)|_{m \to \infty} \longrightarrow 0.$$

The contradiction obtained proves the theorem.

§ 3. For processes with n types of particles, starting with a set $(m_1, \dots, m_n)$ of particles of types $T_1, \dots, T_n$, as $t \to \infty$, $\lambda \to 0$, $m_j \to \infty$ $(j = 1, \dots, n)$, by means of theorem 1 and lemma 7 from ([4]) one can obtain analogues of theorems 3 and 4 from ([4]). The limiting distributions will be concentrated on a one-dimensional straight line, and on this line they will coincide with the limiting distributions of theorems 3 and 4 from ([4]).

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Note: Figure translations are in progress. See original paper for figures.